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Large Deviations for Non-Uniformly Expanding Maps

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We obtain large deviation bounds for non-uniformly expanding maps with non-flat singularities or criticalities and for partially hyperbolic non-uniformly expanding attracting sets. That is, given a continuous function we consider its space average with respect to a physical measure and compare this with the time averages along orbits of the map, showing that the Lebesgue measure of the set of points whose time averages stay away from the space average tends to zero exponentially fast with the number of iterates involved. As easy by-products we deduce escape rates from subsets of the basins of physical measures for these types of maps. The rates of decay are naturally related to the metric entropy and pressure function of the system with respect to a family of equilibrium states.

KEY WORDS: non-uniform expansion, physical measures, hyperbolic times, large deviations.
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Contents

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1. INTRODUCTION

Smooth Ergodic Theory provides asymptotic information on the behavior of a dynamical system, given by a smooth transformation, when times goes to infinity. This statistical approach to Dynamics has provided valuable insights into many phenomena: from the remarkable result of Jakobson(32) (see also Refs. 12, 13) showing the existence of many (positive Lebesgue measure of) parameters $a \in (0, 2)$ for which the corresponding map of the quadratic family $x \mapsto a - x^2$ has positive Lyapunov exponent along almost every orbit; to the study of higher dimensional systems: related ideas provided the first clue to the nature of the Hénon attractor(13,41) or the existence of robust classes of maps which are not uniformly expanding but exhibit several distinct positive Lyapunov exponents,(57) and enabled one to understand the statistical properties of these and other classes of systems.(2,6,14,16,18,45,61)

The basic ideas can be traced back to the Boltzmann Ergodic Hypothesis from Statistical Mechanics which was the main motivation behind the celebrated Birkhoff's Ergodic Theorem ensuring the equality between temporal and spatial averages with respect to a (ergodic) probability measure μ invariant under a measurable transformation $f : M \to M$ of a compact manifold M, i.e. for every continuous map $\varphi : M \to \mathbb{R}$ we have

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi \, d\mu \tag{1}$$

for μ almost every point $x \in M$. Defining $B(\mu)$, the *ergodic basin of* μ, to be the set of points for which (1) holds for every continuous function φ, the Ergodic Theorem says that $\mu(B(\mu)) = 1$ for all ergodic *f*-invariant probability measures μ.

Since ergodic measures can be, for instance, Dirac masses concentrated on periodic orbits, the Ergodic Theorem in itself does not always provide information about the asymptotic behavior of "big" subsets of points. The notion of "big" can arguably be taken as meaning "having positive Lebesgue measure (or positive volume)," since such sets are in principle "observable sets" when interpreting $f : M \to M$ as a model of physical, biological or economic phenomena. Correspondingly invariant probability measures μ for which $B(\mu)$ has positive volume are called *physical* (or Sinai-Ruelle-Bowen) measures.

This kind of measures was first constructed for (uniformly) hyperbolic diffeomorphisms by Sinai, Ruelle and Bowen.[20,49,55] Such measures for non-uniformly hyperbolic maps where obtained more recently.[2,14,15,45]

We say that a local diffeomorphism f of a compact manifold is (uniformly) *expanding* if there exists $n \geq 1$ such that for all x and every tangent vector v at x

$$\|Df^n(x)v\| \geq 2\|v\|. \tag{2}$$

For diffeomorphisms of compact manifolds, the notion of *hyperbolicity* requires the existence of two complementary directions given by two (continuous) subbundles E and F of the tangent bundle admitting $n \geq 1$ such that for all points x and tangent vectors $(u, v) \in E_x \oplus F_x$

$$\|Df^n(x)u\| \leq \frac{1}{2}\|u\| \quad \text{and} \quad \|Df^n(x)v\| \geq 2\|v\|. \tag{3}$$

The probabilistic properties of physical measures are an object of intense study, see e.g. Refs. 3, 5, 7, 10, 16, 20, 29, 61. The leitmotif is that the sequence $\{\varphi \circ f^n\}_{n\geq 0}$ should behave like an i.i.d. random variable, at least asymptotically.

Here we are concerned with the rate of convergence of the time averages (1) for non-uniformly expanding maps (NUE) and partially hyperbolic non-uniformly expanding diffeomorphisms (PHNUE), where condition (2) and the right hand side condition of (3) are replaced by the following asymptotic ones

> **NUE:** for Lebesgue almost every point x there exists $n = n(x) \geq 1$ such that $\|Df^n(x)v\| \geq 2\|v\|$ for all vector $v \in T_xM$;
> **PHNUE:** for Lebesgue almost all points x there exists $n = n(x) \geq 1$ such that $\|Df^n(x)v\| \geq 2\|v\|$ for all vector $v \in F_x$.

We note that if conditions **NUE** or **PHNUE** hold for *every point* then the system is uniformly expanding or uniformly hyperbolic.[4,56] We also consider transformations which are diffeomorphisms outside a "small" (zerovolume) set of singular or critical points such that the orbits of Lebesgue almost all points have slow recurrence near this singular set. For more details see the statement of results below.

The question of the speed of convergence to equilibrium arises naturally from so-called thermodynamical formalism of (uniformly) hyperbolic diffeomorphisms, borrowed from statistical mechanics by Ruelle, Sinai and Bowen (among others, see e.g. Refs. 17, 19, 28, 51, 52) through the dictionary between one-dimensional lattices and (uniformly) expanding maps (Gibbs distributions and equilibrium states in particular) provided by the existence of a finite Markov partition for the latter systems. Indeed chaotic dynamics is associated with loss of memory and creation of information (two views of the same phenomenon) as the system evolves. These notions are formalized in a variety of ways, from *entropy*, the exponential rate of creation of information; to *decay of correlations*, which measures the speed the system "forgets" its initial state; through *large deviations* results, which measure how fast the system approaches a state of equilibrium after evolving from almost every initial state. However, even with abundance of positive Lyapunov exponents, which is the essential content of the non-uniform expansion/hyperbolicity conditions above, extending this theory from uniform to the non-uniform hyperbolic setting demands considering (if one is optimistic), through the dictionary already mentioned, Markov partitions with infinitely many symbols leading to a thermodynamical formalism of gases with infinitely many states, a hard subjects not yet well understood (see e.g. Refs.10, 23 for recent developments).

Assuming conditions **NUE** or **PHNUE** we are able to extend some of the large deviation results for uniformly hyperbolic system in Refs. 34, 60 (see also Refs. 26, 27 for sharp estimates though a different approach) and strengthen, in a definite sense, the idea that non-uniformly hyperbolic systems are *chaotic:* they satisfy a version of the classical large deviation results for i.i.d. random variables. More precisely, if we set $\delta > 0$ as an acceptable error margin and consider

$$B_n = \left\{ x \in M : \left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) - \int \varphi d\mu \right| > \delta \right\}$$

then we are able to ascertain whether the Lebesgure measure of B_n decays to zero exponentially fast, i.e. weather there are constants $C, \xi > 0$ such that

$$\text{Leb}(B_n) \le Ce^{-\xi n} \quad \text{for all} \quad n \ge 1. \tag{4}$$

The values of $C, \xi > 0$ above depend on δ, φ and on global invariants for the map f such as the metric entropy and the pressure function of f with respect to some equilibrium measures, as detailed in the next section.

We are able to obtain large deviation rates as in (4) for non-uniformly expanding local diffeomorphisms and also for endomorphisms and maps with non-flat singularities and critical points under a condition on the rate of approximation of most orbits to the critical/singular set. In particular we are able to obtain an exponential decay rate as above for piecewise expanding maps with infinitely many smoothness domains, for quadratic maps corre-sponding to a positive Lebesgue measure subset of parameters and for a class of maps with infinitely many critical points. Moreover we also ob-tain the same kind of rates for partially hyperbolic attracting sets with a non-uniformly expanding direction.

1.1. Statement of the Results

We denote by $\|\cdot\|$ a Riemannian norm on the compact boundaryless manifold M, by d the induced distance and by Leb a Riemannian volume form, which we call *Lebesgue measure* or *volume* and assume to be normalized: $\mathrm{Leb}(M) = 1$.

We start by describing one of the class of maps that we are going to consider. Let $f : M \to M$ be a map of the compact manifold M which is a C^2 local diffeomorphism outside a set $\mathcal{S} \subset M$ with zero Lebesgue measure. We assume that *f behaves like a power of the distance* close to $\mathcal{S}$: there are constants $B > 1$ and $\beta > 0$ for which

(S1) $\frac{1}{B} d(x,\mathcal{S})^{\beta} \le \frac{\|Df(x)v\|}{\|v\|} \le B d(x,\mathcal{S})^{-\beta}$;
(S2) $|\log\|Df(x)^{-1}\| - \log\|Df(y)^{-1}\|| \le B\frac{D(x,y)}{d(x,\mathcal{S})^{\beta}}$;
(S3) $|\log|\det Df(x)^{-1}| - \log|\det Df(y)^{-1}|| \le B\frac{d(x,y)}{d(x,\mathcal{S})^{\beta}}$;

for every $x, y \in M\setminus\mathcal{S}$ with $d(x,y) < d(x,\mathcal{S})/2$ and $v \in T_xM\setminus\{0\}$. The singular set $\mathcal{S}$ may be thought of as containing those points x where $Df(x)$ is either not defined or else is non-invertible. Note in particular that $\mathcal{S}$ contains the set $\mathcal{C}$ of critical points of f, *i.e.* the set of points (which may be empty) where $Df(x)$ is not invertible. We refer to this kind of singular sets as non-flat since conditions (S1) to (S3) above are natural generalizations to arbitrary dimensions of the notion of non-flat critical point from one-dimensional dynamics, see e.g. Ref. 25.

In what follows we write $S_n\varphi(x)$ for $\sum_{i=0}^{n-1}\varphi(f^i(x))$ and a function $\varphi : M \to \mathbb{R}$. We say that f as above is *non-uniformly expanding* if there exists $c > 0$ such that

$$\limsup_{n\to+\infty}\frac{1}{n}S_n\psi(x) \le -c \quad \text{where} \quad \psi(x) = \log\|Df(x)^{-1}\|, \tag{5}$$

for Lebesgue almost every $x \in M$. We need to control the rate of approximation of most orbits to the singular set. We say that f has *slow recurrence to the singular set* $\mathcal{S}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_{n\to\infty}\frac{1}{n}S_n\Delta_\delta(x) < \varepsilon \quad \text{with} \quad \Delta_\delta(x) = |\log d_\delta(x,\mathcal{S})| \tag{6}$$

for Lebesgue almost every $x \in M$, where for any given $\delta > 0$ we define the *smooth δ-truncated distance* from $x \in M$ to $\mathcal{S}$ by

$$d_\delta(x,\mathcal{S}) = \xi_\delta(d(x,\mathcal{S}))\cdot d(x,\mathcal{S}) + 1 - \xi_\delta(d(x,\mathcal{S}))$$

where $\xi_\delta : \mathbb{R} \to [0, 1]$ is a standard C^∞ auxiliary function satisfying

$$\xi_\delta(x) = 1 \text{ if } |x| \le \delta \quad \text{and} \quad \xi_\delta(x) = 0 \text{ if } |x| \ge 2\delta.$$

Observe that Δ_δ is non-negative and continuous away from $\mathcal{S}$ and identically zero 2δ-away from $\mathcal{S}$.

These notions where presented in Ref. 6 for higher dimensional maps abstracted from similar notions from one-dimensional maps[25] and previous work on maps with singularities,[33] and in Refs. 1, 6, the following result on existence of finitely many physical measures was obtained.

Theorem 1.1. *Let $f : M \to M$ be a C^2 local diffeomorphism outside a nonflat singular set $\mathcal{S}$. Assume that f is non-uniformly expanding with slow recurrence to $\mathcal{S}$. Then there are finitely many physical (or* Sinai-Ruelle-Bowen*) measures $\mu_1, \dots, \mu_k$ whose basins cover the manifold Lebesgue almost everywhere, that is $B(\mu_1) \cup \dots \cup B(\mu_k) = M$,* Leb − mod 0.

We say that f is a *regular map* if $f_*\text{Leb} \ll \text{Leb}$, that is, if $E \subset M$ is such that $\text{Leb}(E) = 0$, then $\text{Leb}(f^{-1}(E)) = 0$. We denote by $\mathcal{M}_f$ the family of all invariant probability measures with respect to *f*, by $\mathcal{M}_f^e$ the family of all *ergodic f*-invariant probability measures, and define

$$B(x, n, \varepsilon) = \{y \in M : d(f^i(x), f^i(y)) < \varepsilon, \qquad i = 0, \dots, n-1\}$$

the (n, ε)-dynamical ball around $x \in M$. Large deviation statements are usually related to *local entropies* which originated from the works of Shannon, McMillan and Breiman[21,39,54] and can be succinctly expressed as follows on a metric space after the work of Brin and Katok.[22] For any finite Borel measure m on M define its local entropy at x to be

$$h_m(f)(x) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log m\left(B(x, n, \varepsilon)\right).$$

In Ref. 22 it is proved that this limit exists m-almost everywhere whenever m is a *f*-invariant probability measure. The metric (or measure-theoretic) entropy of the map *f* is then defined to be the non-negative number

$$h_m(f) = \int h_m(f)(x)\, dm(x).$$

Moreover the function $h_m(f)(x)$ is *f*-invariant, so it is almost everywhere constant if m is *f*-ergodic.

We will be interested in the case m = Lebesgue measure (volume) on M, which is usually *not* an invariant measure in our setting and for $\nu \in \mathcal{M}_f$ we consider

$$h_m(f, \nu) = \nu - \operatorname{ess\,sup} h_m(f).$$

Note that given $\nu \in \mathcal{M}_f$ the value of $h_\nu(f)$ is not at all related to $h_{\mathrm{Leb}}(f, \nu)$, unless both measures coincide and $\nu \in \mathcal{M}_f^e$, in which case $h_\nu(f, \nu) = h_\nu(f)$.

Theorem A. *Let $f : M \to M$ be a regular $C^{1+\alpha}$ local diffeomorphism outside a non-flat singular set $\mathcal{S}$, for some $\alpha \in (0, 1)$. Assume that f is non-uniformly expanding with slow recurrence to $\mathcal{S}$. Then writing $J = \log|\det Df|$, given $c \in \mathbb{R}$ and a continuous function $\varphi : M \to \mathbb{R}$*

(1) if $h_{\mathrm{top}}(f) < \infty$, then

$$\liminf_{n\to+\infty} \frac{1}{n} \log \mathrm{Leb}\left(\left\{x \in M : \frac{1}{n} S_n\varphi(x) > c\right\}\right) \geq \sup\left\{h_\nu(f) - h_{\mathrm{Leb}}(f, \nu) : \nu \in \mathcal{M}_f^e, \int \varphi d\nu > c\right\};$$

(2) if $\mathcal{S} = \emptyset$ (f is a local diffeomorphism) then

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathrm{Leb}\left(\left\{x \in M : \frac{1}{n} S_n\varphi(x) \geq c\right\}\right) \leq \sup\left\{h_\nu(f) - \int J d\nu : \nu \in \mathcal{M}_f, \int \varphi d\mu \geq c\right\}.$$

(3) in general for any given $\eta > 0$ there exists $\varepsilon, \delta > 0$ such that

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathrm{Leb}\left(\left\{x \in M : \frac{1}{n} S_n\varphi(x) \geq c \quad \textit{and} \quad \frac{1}{n} S_n \Delta_\delta(x) \leq \varepsilon\right\}\right) \leq \eta + \sup\left\{h_\nu(f) - \int J d\nu : \nu \in \mathcal{M}_f, \int \varphi d\nu \geq c \quad \textit{and} \quad \Delta_\delta \in L^1(\nu)\right\}.$$

We say that a measure $\upsilon \in \mathcal{M}_f$ is an *equilibrium state for f with respect to J* (or just an *equilibrium state* in what follows) if

$$h_\nu(f) = \nu(J) = \int J d\nu.$$

As the above statement shows, equilibrium states are involved in the determination of the asymptotic rates of deviation. Given $\varepsilon, \delta > 0$ we write $\mathbb{E} = \mathbb{E}_{\varepsilon,\delta}$ for the family of all equilibrium states μ of f with respect to J such that $\mu(\Delta_\delta) \leq \varepsilon$ and, given a continuous $\varphi : M \to \mathbb{R}$, we define $\mathbb{E}(\varphi) = \{\nu(\varphi) : \nu \in \mathbb{E}\}$.

Remark 1.2. Note that the expressions obtained in items (1) and (2) of the statement of Theorem A are *not comparable* since the supremum is taken over all invariant measures in item (2), while we consider only ergodic invariant measures in item (1).

From Theorem A we are able to deduce that the supremum above is strictly negative for non-uniformly expanding maps with slow recurrence to the singular set.

Theorem B. *Let $f : M \to M$ be a local diffeomorphism outside a non-flat singular set $\mathcal{S}$ which is non-uniformly expanding and has slow recurrence to $\mathcal{S}$. For $\omega > 0$ and a continuous function $\varphi : M \to \mathbb{R}$ there exists $\varepsilon, \delta > 0$ arbitrarily close to 0 such that, writing*

$$A_n = \left\{ x \in M : \frac{1}{n} S_n \Delta_\delta(x) \geq \varepsilon \right\}$$

and

$$B_n = \left\{ x \in M : \inf \left\{ \left| \frac{1}{n} S_n\varphi(x) - \eta(\varphi) \right| : \eta \in \mathbb{E} \right\} > \omega \right\} \tag{7}$$

we get

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathrm{Leb}(A_n \cap B_n) < 0. \tag{8}$$

Clearly if $\mathcal{S} = \emptyset$ (f is a local diffeomorphism) then $A_n = M$ and we obtain an asymptotic large deviation rate for the sets B_n. Otherwise to get a similar upper bound for $\mathrm{Leb}(B_n)$ we need an extra assumption on the decay of the measure of the *tail sets* $M \setminus A_n$.

Corollary C. *In the setting of Theorem B with $\mathcal{S} \neq \emptyset$, if f also satisfies*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathrm{Leb}(M \setminus A_n) < 0 \tag{9}$$

then we have also

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathrm{Leb}(B_n) < 0.$$

Remark 1.3. Observe that if μ is a f-ergodic absolutely continuous probability measure whose support is the entire manifold, then the slow recurrence condition (6) is the same as saying that $\log d(x, \mathcal{S})$ is μ-integrable.

Note that for any C^2 *endomorphism f* (i.e. the singular set $\mathcal{S}$ of *f* coincides with the critical set $\mathcal{S}$ of f) we have $|\log d(x, \mathcal{C})| \geq \Delta_\delta(x)$ and, as shown in Ref. 36, the function $\log d(x, \mathcal{C})$ is μ-integrable for every f-invariant probability measure. However we need to deal with families of invariant probability measures for which $\log d(x, \mathcal{S})$ is *uniformly integrable* so that the proofs of Theorems A and B can be carried out with our arguments. This is why we need the sets A_n in the previous statements. To the best of our knowledge no such general integrability result for $\log d(x, \mathcal{S})$ exists with respect to invariant probability measures for maps with non-flat singularities.

1.2. Partially Hyperbolic Diffeomorphisms

Let now $f : M \to M$ be a C^2 diffeomorphism. We say that a compact f-invariant set Λ is an *attracting set* if it admits a *trapping region*, that is, an open neighborhood $U \supset \Lambda$ such that $\overline{f(U)} \subset U$ and $\Lambda = \cap_{n\geq 0} f^n(U)$. Note that we may have $\Lambda = U = M$ (where M is connected).

As shown in Ref. 60, for every attracting set Λ and every physical probability measure ν supported in Λ, given $\delta > 0$ and a continuous $\varphi : \overline{U} \to \mathbb{R}$ we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \text{Leb} \left\{ \left| \frac{1}{n} S_n\varphi - \int \varphi d\mu \right| > \delta \right\} \geq$$
$$\sup \left\{ h_\nu(f) - \int \Sigma^+ d\nu : \nu \in \mathcal{M}^e_f, \left| \int \varphi d\nu - \int \varphi d\mu \right| \geq \delta \right\}.$$

Here Σ^+ denotes the sum of the positive Lyapunov exponents at a given point of M. Recall that Ruelle's Inequality $h_\mu(f) \leq \int \Sigma^+ d\mu$ is true of every C^1-diffeomorphism.[50]

An attracting set Λ is *partially hyperbolic* (see e.g. Refs. 17, 45) if there exists a continuous splitting $E \oplus F$ of the tangent bundle of M over Λ along two complementary vector subbundles satisfying

- Df-invariance: $Df(E_x) = E_{f(x)}$ and $Df(F_x) = F_{f(x)}$ for all $x \in \Lambda$;
- domination: there exists $n \geq 1$ such that

$$\|Df^n|E_x\| \cdot \|(Df^n|F_x)^{-1}\| \leq \frac{1}{2} \quad \text{for all} \quad x \in \Lambda;$$

- E is uniformly contracting: there is $n \geq 1$ such that $\|Df^n|E_x\| \leq \frac{1}{2}$ for all $x \in \Lambda$.

In this setting we denote by J the logarithm of the Jacobian along the centre-unstable direction $J(x) = \log|\det Df \mid F_x|$ and by $\mathbb{E}$ the family of all *equilibrium states* with respect to J, i.e. the set of all f-invariant probability measures υ such that $h_\nu(f) = \nu(J)$.

We will assume further that the F direction only has positive Lyapunov exponents in the following sense, introduced in Ref. 6. We say that a partially hyperbolic attractor with trapping region U is *non-uniformly expanding* if there exists $c > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|\left(Df|F_{f^j(x)}\right)^{-1}\| \leq -c$$

for Lebesgue almost every point $x \in U$. In Ref. 6 the following was obtained.

Theorem 1.4. *Let Λ be a partially hyperbolic non-uniformly expanding attracting set for a C^2 diffeomorphism f with trapping region U. Then there are finitely many equilibrium states which are physical measures supported in Λ, and whose basins cover U except for a subset of zero Lebesgue measure.*

We are able to obtain an upper bound entirely analogous to item 2 of Theorem A replacing M by the points in the trapping region U of a partially hyperbolic non-uniformly expanding attracting set Λ for a C^2 diffeomorphism. Then for the same kind of attracting sets we obtain an upper bound for the subset corresponding to (7).

Theorem D. *Let $f : M \to M$ be a C^2 diffeomorphism exhibiting a partially hyperbolic non- uniformly expanding attracting set Λ with isolating neighborhood $U \supset \Lambda$. Given $\omega > 0$ and a continuous $\varphi : \overline{U} \to \mathbb{R}$, define*

$$B_n = \left\{x \in U : \inf\left\{\left|\frac{1}{n}S_n\varphi(x) - \eta(\varphi)\right| : \eta \in \mathbb{E}\right\} > \omega\right\}.$$

Then

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathrm{Leb}(B_n) < 0.$$

1.3. Escape Rates

Using the estimates obtained above and the observation that for any compact subset K and a given $\varepsilon > 0$ we can find an open set $W \supset K$ and a continuous function $\varphi : M \to \mathbb{R}$ such that

- $\mathrm{Leb}(W \backslash K) < \varepsilon$;
- $0 \leq \varphi \leq 1$, $\varphi|K \equiv 1$ and $\varphi|(M\backslash W) \equiv 0$,

we see that for $n \geq 1$

$$\{x \in K : f(x) \in K, \ldots, f^{n-1}(x) \in K\} \subset \left\{x \in M : \frac{1}{n}S_n\varphi(x) \geq 1\right\} \tag{10}$$

and so we get the following (recall the definition of A_n in the statement of Theorem B).

Corollary E. *Let $f : M \to M$ be a local diffeomorphism outside a non-flat singular set $\mathcal{S}$ which is non-uniformly expanding and has slow recurrence to $\mathcal{S}$. Let K be a compact subset such that $\mu(K) < 1$ for all μ, in the weak*-closure $\bar{\mathbb{E}}$ of $\mathbb{E}$. Then for a pair $\varepsilon, \delta > 0$ close to 0*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \text{Leb}\left(\{x \in K \cap A_n : f^j(x) \in K, j = 1, \dots, n-1\}\right) < 0.$$

Moreover if $\limsup_{n\to\infty} \frac{1}{n} \log \text{Leb}(M \setminus A_n) < 0$ *then*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \text{Leb}\left(\{x \in K, f(x) \in K, \dots, f^{n-1}(x) \in K\}\right) < 0.$$

In the setting of a partially hyperbolic non-uniformly expanding attracting set we get, using the same reasoning as above

Corollary F. *Let $f : M \to M$ be a diffeomorphism and Λ a partially hyperbolic non-uniformly expanding attracting set with isolating neighborhood U. Let $K \subset U$ be a compact subset such that $\mu(K) < 1$ for all μ in the weak*-closure $\bar{\mathbb{E}}$ of $\mathbb{E}$. Then*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \text{Leb}\left(\{x \in K, f(x) \in K, \dots, f^{n-1}(x) \in K\}\right) < 0.$$

1.4. Comments and Organization of the Paper

All the arguments use in fact that f is C^1 and that its derivative Df is α-Hölder continuous with respect to the fixed Riemannian norm on M, so that all we need is f to be a $C^{1+\alpha}$ local diffeomorphism outside the singular set, for some $\alpha \in (0, 1)$.

The difficulties we face when considering transformations which are not uniformly hyperbolic and present singularities are related to the construction of the measures ν, appearing in the supremum at item (1) of the statement of Theorem A, as a weak* limit of discrete measures which converge to an invariant measure and are supported on the set one wishes to control. Since we need to take weak* limits of measures against discontinuous test functions, the main body of work in this paper is to provide sufficient estimates for convergence imposing some conditions on the dynamics of the maps involved.

The existence of a lower bound for the large deviation rate with the same expression as in items (2) and (3) of the statement of Theorem A depends on the existence and uniqueness of equilibrium states (the reader should see Ref. 34 for precise statements and also for counter-examples when uniqueness is not satisfied). However existence and uniqueness of equilibrium states for non-uniformly expanding maps is still an open problem for most potentials in spite of recent progress in this direction by several authors, see e.g. Refs. 10, 11, 43.

Recently Pinheiro[46] has extended the statement of Theorem 1.1 replacing the limsup in condition (5) by liminf, keeping the same conclusions involving the existence of finitely many physical measures and of a positive density of hyperbolic times Lebesgue almost everywhere. Hence our statements are automatically valid in this more general setting.

In what follows, we start by presenting some non-trivial classes of maps to which our results are applicable, in Sec. 2. In Sec. 3 we present preliminary technical results to be used in the following sections. Theorem A is then proved in Sec. 4.1 for local diffeomorphisms, in Sec. 4.2 for partially hyperbolic non-uniformly expanding diffeomorphisms and in Sec. 4.3 for maps with singularities or criticalities. In Sec. 5 we deduce Theorem B from Theorem A, first for local diffeomorphisms and for the partially hyperbolic case in Sec. 5.1, and then with singularities or criticalities in Sec. 5.2, together with an extension of Ruelle's Inequality to maps with non-flat singularities in Sec. 5.3.

2. EXAMPLES OF APPLICATION

Here we show that there are many examples of maps in the conditions of Theorem B, Corollary C or Theorem D.

2.1. Quadratic Maps and Infinite-Modal Maps

In Ref. 8 the following C^∞ family of maps of $I = [-1, 1]$ with infinitely many critical points was considered:

$$f_\mu(z) = \begin{cases} f(z) + \mu & \text{for } z \in (0, \varepsilon] \\ f(z) - \mu & \text{for } z \in [-\varepsilon, 0) \end{cases}$$

where $f : I \to I$ is an expanding extension of

$$\hat{f} : [-\varepsilon, \varepsilon] \to [-1, 1], \quad \hat{f}(z) = \begin{cases} az^\alpha \sin(\beta \log(1/z))) & \text{if } z > 0 \\ -a|z|^\alpha \sin(\beta \log(1/|z|))) & \text{if } z < 0 \end{cases},$$

to I (i.e. $|f'| \gg 1$ on $I \setminus [-\varepsilon, \varepsilon]$), with $a > 0$, $0 < \alpha < 1$, $\beta > 0$ and $\varepsilon > 0$. It was shown that there exists a positive Lebesgue measure subset P of parameters in $(-\varepsilon, \varepsilon)$ such that for $\mu \in P$ the map f_μ is non-uniformly expanding and has slow recurrence to the non-flat infinite and denumerable singular set. Moreover for the same parameters the decay rate of the tail set is exponential, i.e. (9) is true and hence f_μ for $\mu \in P$ is in the setting of Corollaries C and E.

Analogous results hold for the quadratic family $Q_a(x) = a - x^2$ (and also for general C^2 unimodal families), so that Corollaries C and E apply to quadratic maps for a positive Lebesgue measure subset of parameters.

2.2. Piecewise Smooth One-Dimensional Expanding Maps

Let $f : I \to I$ be a map admitting a sequence $\mathcal{S} = \{a_n, n \geq 1\} \subset I = [-1, 1]$ such that for every connected component G of $I \backslash \mathcal{S}$ we have that $f|G$ is C^1 diffeomorphism with its image. Assume that $\mathcal{S}$ is a non-flat singular set for f and that f admits a absolutely continuous ergodic invariant probability measure μ with positive Lyapunov exponent and such that $\log d(x, \mathcal{S})$ is μ-integrable and $\mathrm{supp}\mu = I$. Then f is in the setting of Theorem B.

Examples of this kind of maps are the Gauss map,[58] and transitive piecewise one dimensional maps satisfying the conditions in Ref. 53 (see also Ref. 58), that is there exists $\kappa > 0$ such that for every connected component G of $I \backslash \mathcal{S}$ we also have

$$\mathrm{var}_G \frac{1}{|f'|} \leq \kappa \cdot \sup_G \frac{1}{|f'|} \quad \text{and} \quad \sum_G \sup_G \frac{1}{|f'|} \leq \kappa.$$

More concrete examples are Lorenz-like maps[35,58] (even with criticalities[37]) and the maps introduced by Rovella.[40,48]

A proof of the exponential decay of the tail set for this class of maps is not available in the literature to the best of our knowledge but can be done as an application of the technique of exclusion of parameters introduced in Ref. 12 (the details will appear in forthcoming work), so that Corollaries C and E also hold for this type of maps.

2.3. Non-Uniformly Expanding Local Diffeomorphisms

Consider a local diffeomorphism $f : M \to M$, so that $\mathcal{S} = \emptyset$, which satisfies

- $\|(Df)^{-1}\| \leq 1$ and
- $K_1 = \{x \in M : \|Df(x)^{-1}\| = 1\}$ is finite.

Then by the results in Ref. 9 we have that such f has a finite set $\mathbb{E}$ of equilibrium states for ϕ. Hence in this case Theorem B holds for every continuous function $\varphi : M \to \mathbb{R}$.

2.4. Viana Maps

The following family of endomorphisms of the cylinder was introduced by Viana. Ref. 57. Let $a_0 \in (1, 2)$ be such that the critical point $x = 0$ is preperiodic for the quadratic map $Q(x) = a_0 - x^2$. Let $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and $b : \mathbb{S}^1 \to \mathbb{R}$ be a Morse function, for instance $b(s) = \sin(2\pi s)$. For fixed small $\alpha > 0$, consider

$$\hat{f} : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$$
$$(s, x) \to (\hat{g}(s), \hat{q}(s, x))$$

where $\hat{g}$ is the uniformly expanding map of the circle defined by $\hat{g}(s) = d \cdot s \pmod{\mathbb{Z}}$ for some $d \geq 16$, and $\hat{q}(s, x) = a(s) - x^2$ with $a(s) = a_0 + \alpha b(s)$. For $a > 0$ small enough there exists an interval $I \subset (-2, 2)$ such that $\hat{f}(S^1 \times I)$ is contained in the interior of $S^1 \times I$. Hence any map f sufficiently C^0 close to $\hat{f}$ has $S^1 \times I$ as a forward invariant region. We consider from here on these maps f close to $\hat{f}$ restricted to $\mathbb{S}^1 \times I$.

In Refs. 2, 3, 57 a C^3 neighborhood $\mathcal{U}$ of $\hat{f}$ was studied and it was proved that every $f \in \mathcal{U}$ is non-uniformly expanding and has slow recurrence to the non-flat critical set $\mathcal{C}$. The arguments in Ref. 57 where extended in Ref. 24 to encompass the weaker condition $d \geq 2$ on the expansion of $\hat{g}$, providing the same properties for a C^∞ -neighborhood $\tilde{\mathcal{U}}$ of $\hat{f}$.

Hence, each $f \in \tilde{\mathcal{U}}$ or $f \in \mathcal{U}$ is in the setting of Theorem B. Results in Refs. 7, 29 show that the tail set decays at least sub-exponentially fast, which is not enough to ensure that Corollaries C and E are true for the maps in $\mathcal{U} \cup \tilde{\mathcal{U}}$. It is conjectured that the tail set indeed decays exponentially fast and with a uniform rate for all maps in $\mathcal{U} \cup \tilde{\mathcal{U}}$.

2.5. Partially Hyperbolic Non-Uniformly Expanding Diffeomorphisms

We sketch the construction of a robust class of partially hyperbolic nonuniformly expanding diffeomorphisms, taking U equal to M, following.[6] This construction is closely related to the C^1 open classes of transitive non-Anosov diffeomorphisms presented in Ref. 18, Sec. 6, as well as other robust examples from Ref. 38.

Start with a linear Anosov diffeomorphism $\hat{f}$ on the d-dimensional torus $M = \mathbb{T}^d$, $d \geq 2$. Write $TM = E \oplus F$ the corresponding hyperbolic decomposition of the tangent bundle. Let V be a small closed domain in M for which there exist unit open cubes K^0 and K^1 in $\mathbb{R}^d$ such that $V \subset \pi(K^0)$ and $\hat{f}(V) \subset \pi(K^1)$, where $\pi : \mathbb{R}^d \to \mathbb{T}^d$ is the canonical projection. Let now f be a diffeomorphism on $\mathbb{T}^d$ such that

(A) f admits invariant cone fields C_E and C_F, with small width $a > 0$ and containing, respectively, the stable bundle E and the unstable bundle F of $\hat{f}$;

(B) f is *partially hyperbolic and volume expanding along the centerunstable direction:* there is $\sigma_1 > 1$ so that

$$|\det (Df \mid T_x\mathcal{D}_F)| > \sigma_1 \quad \text{and} \quad \|Df \mid T_x\mathcal{D}_E\| < \sigma_1^{-1}$$

for any $x \in M$ and any disks $\mathcal{D}_F$, $\mathcal{D}_E$ tangent to C_F, C_E, respectively (see Sec. 3.2 for more on invariant cone fields and disks tangent to cone fields in this setting).

(C) f is C^1-close to $\hat{f}$ in the complement of V, so that there exists $\sigma_2 < 1$ satisfying

$$\|(Df \mid T_x\mathcal{D}_f)^{-1}\| < \sigma_2 \quad \text{and} \quad \|Df \mid T_x\mathcal{D}_E\| < \sigma_2$$

for any $x \in (M \backslash V)$ and any disks $\mathcal{D}_F, \mathcal{D}_E$ tangent to C_F, C_E, respectively. Moreover $f(V)$ is also contained in the projection of a unit open cube.

(D) there exist some small $\delta_0 > 0$ satisfying

$$\|(Df|T_x\mathcal{D}_F)^{-1}\| < 1 + \delta_0$$

for any $x \in V$ and any disk $\mathcal{D}_F$ tangent to C_F.

If $\tilde{f}$ is a torus diffeomorphism satisfying (A), (B), (D), and coinciding with $\hat{f}$ outside V, then any map f in a C^1 neighborhood of $\tilde{f}$ satisfies all the previous conditions. Results In Ref. (6, Appendix) show in particular that for any f satisfying (A)–(D) there exist $c_u > 0$ such that f is partially hyperbolic and non-uniformly expanding along its center-unstable direction, as defined in Sec. 1.2. Hence on a small C^2 neighborhood $\mathcal{U}$ of $\tilde{f}$ every diffeomorphism $f \in \mathcal{U}$ satisfies all the conditions of Theorem D.

3. HYPERBOLIC TIMES

The main technical tool used in the study of non-uniformly expanding maps is the notion of hyperbolic times, introduced in Refs. 2, 47. We say that n is a (σ, δ, b)- hyperbolic time of f for a point x if the following two conditions hold with $0 < \sigma < 1$ and $b, \delta > 0$

$$\prod_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| \le \sigma^k \quad \text{and} \quad d_\delta(f^k(x), \mathcal{S}) \le e^{-bk} \tag{11}$$

for all $k = 0, \ldots, n-1$.

We now outline the properties of these special times. For detailed proofs see (Ref. 6, Proposition 2.8) and (Ref. 3, Proposition 2.6, Corollary 2.7, Proposition 5.2).

Proposition 3.1. *There are constants $C_1, \delta_1 > 0$ depending on (σ, δ, b) and f only such that, if n is (σ, δ, b)-hyperbolic time of f for x, then there are hyperbolic preballs $V_k(x)$ which are neighborhoods of $f^{n-k}(x)$, $k = 1, \ldots, n$, such that*

(1) $f^k|V_k(x)$ maps $V_k(x)$ diffeomorphically to the ball of radius δ_1 around $f^n(x)$;

(2) for every $1 \le k \le n$ and $y, z \in V_k(x)$

$$d(f^{n-k}(y), f^{n-k}(z)) \le \sigma^{k/2} \cdot d(f^n(y), f^n(z));$$

(3) for $y, z \in V_k(x)$

$$\frac{1}{C_1} \leq \frac{|\det Df^{n-k}(y)|}{|\det Df^{n-k}(z)|} \leq C_1.$$

The following ensures existence of infinitely many hyperbolic times Lebesgue almost every point for non-uniformly expanding maps with slow recurrence to the singular set. A complete proof can be found in Ref. 6, Sec. 5.

Theorem 3.2. *Let* $f : M \to M$ *be a* $C^{1+\alpha}$ *local diffeomorphism away from a non-flat singular set* $\mathcal{S}$*, for some* $\alpha \in (0, 1)$*, non-uniformly expanding and with slow recurrence to* $\mathcal{S}$*. Then there are* $\sigma \in (0, 1)$*,* $\delta, b > 0$ *and there exists* $\theta = \theta(\sigma, \delta, b) > 0$ *such that Leb-a.e.* $x \in M$ *has infinitely many* (σ, δ, b)*-hyperbolic times. Moreover if we write* $0 < n_1 < n_2 < n_2 < \dots$ *for the hyperbolic times of x then their asymptotic frequency satisfies*

$$\liminf_{N \to \infty} \frac{\#\{k \geq 1 : n_k \leq N\}}{N} \geq \theta \quad \textit{for} \quad \text{Leb-}a.e.\ x \in M.$$

3.1. Coverings by Hyperbolic Preballs

Lemma 3.3. *Let* $B \subset M, \theta > 0$ *and* $g : M \to M$ *be a local diffeomorphisms outside a non-flat exceptional set* $\mathcal{S}$ *such that g has density* $> 2\theta$ *of hyperbolic times for every* $x \in B$*. Then, given any probability measure* ν *on B and any* $m \geq 1$*, there exists* $n > m$ *such that*

$$\nu(\{x \in B : n \textit{ is a hyperbolic time of } g \textit{ for } x\}) > \frac{\theta}{2}.$$

This is (Ref. 43, Lemma 4.4) easily adapted to our setting. For completion we include its very short proof. This lemma shows that we can translate the density of hyperbolic times into the Lebesgue measure of the set of points which have a specific (large) hyperbolic time.

Proof: Let H be the set of pairs $(x, n) \in B \times \mathbb{N}$ for which n is a hyperbolic time of g for x. For each $k \geq 1$, let $\#_k$ be the normalized counting measure on $\{m + 1, m + 2, \dots, m + k\}$. Our assumption implies that for any given $x \in B$ we have for big enough $k \geq 1$

$$h_k(x) = \#_k(\pi(H \cap (\{x\} \times \mathbb{N}))) > 2\theta,$$

where $\pi : B \times \mathbb{N} \to \mathbb{N}$ is the projection on the second coordinate. Given any probability measure ν on B we have by Fatou's Lemma

$$\liminf_{k \to \infty} \int h_k d\nu \geq \int \liminf_{k \to \infty} h_k d\nu \geq 2\theta$$

so we may fix $k \geq 1$ large enough so that $\nu(h_k) > \theta$ and find a subset for $C \subset B$ with $\nu(C) > 1/2$ and $h_k(x) \geq \theta/2$ for all $x \in C$. By Fubini's Theorem this means that

$$(\nu \times \#_k)(H) > \theta \text{ and thus } \nu(\hat{\pi}(H \cap (B \times \{n\}))) > \frac{\theta}{2}$$

for some $m < n \leq m + k$, where $\hat{\pi} : B \times \mathbb{N} \to B$ is the projection on the first coordinate. This proves the lemma. $\square$

Let f be a regular map in the setting of the Main Theorem with positive density of (σ, δ)-hyperbolic times Lebesgue almost everywhere. Let $\mathcal{E} = \{B(x_i, \delta_1/8), i = 1, \ldots, l\}$ be a finite open cover of M by $\delta_1/8$- balls. From this we define a finite partition $\mathcal{P}$ of M as follows. We start by setting $P_1 = B(x_1, \delta_1/8)$ as the first element of the partition. Then, assuming that $P_1, \ldots, P_k$ are already defined we set $P_{k+1} = B(x_{k+1,}\delta_1/8)\backslash(P_1 \cup \ldots \cup P_k)$ for $k = 1, \ldots, l-1$. Note that if $P_k \neq \emptyset$ then P_k has non-empty interior, diameter smaller than $\delta_1/4$ and the boundary ∂P_k is a (finite) union of pieces of boundaries of balls in a Riemannian manifold, thus has zero Lebesgue measure. We define $\mathcal{P}$ by the elements P_k constructed above which are non-empty.

Note that since f is regular the boundary of $g(P)$ still has zero Lebesgue measure for every atom $P \in \mathcal{P}$ and every inverse branch g of f^n, for any $n \geq 1$.

Let us choose one interior point in each atom $P \in \mathcal{P}$ and form the set $\mathcal{C}_0$ of representatives of the atoms of $\mathcal{P}$. Let $d_0 = \min\{d(w, \partial\mathcal{P}), w \in \mathcal{C}_0\} > 0$ where $\partial\mathcal{P} = \cup_{P\in\mathcal{P}}\partial\mathcal{P}$ is the boundary of $\mathcal{P}$.

Lemma 3.4. *Let $(\mu_n)_{n\geq 1}$ be a family of Borel probability measures on M and μ some weak* accumulation point of the sequence (μ_n). Then given $0 < \varepsilon < d_0$ there exists a partition $\mathcal{P}_\varepsilon$ with the same number of atoms of $\mathcal{P}$, whose atoms have non-empty interior, diameter smaller than $\delta_1/2$ and whose boundaries have zero Lebesgue measure, such that*

(1) $\mu(\partial\mathcal{P}_\varepsilon) = 0$ and $\mu_n(\partial\mathcal{P}_\varepsilon = 0$ for all $n \geq 1$;
(2) each $P \in \mathcal{P}_\varepsilon$ contains one, and only one, element of $\mathcal{C}_0$;
(3) given $\delta > 0$ we may find $0 < \varepsilon < \min\{\delta, d_0\}$ such that for each $P \in \mathcal{P}_\varepsilon$ there is $Q \in \mathcal{P}$ satisfying Leb $(P\Delta Q) < \epsilon < \delta\cdot$ Lab *(Q).*

Proof: Let us take $0 < \gamma < \min\{\varepsilon, \delta_1/8\}$ such that for all $i = 1, \ldots, l$

$$\text{Leb}\left(B\left(x_i, \frac{\delta_1}{8} + \gamma\right)\Big\backslash B\left(x_i, \frac{\delta_1}{8}\right)\right) < \frac{\varepsilon}{l} \tag{12}$$

and also for all $n \geq 1$

$$\mu\left(\partial B\left(x_i, \frac{\delta_1}{8} + \gamma\right)\right) = 0 = \mu_n\left(\partial B\left(x_i, \frac{\delta_1}{8} + \gamma\right)\right). \tag{13}$$

Such value of γ exists since the set of values of $\gamma > 0$ such that some of the expressions in (13) is positive for some $i \in \{1, \dots, l\}$ and some $n \geq 1$ is denumerable. Thus we may take $\gamma > 0$ satisfying (13) arbitrarily close to zero, and so inequality (12) can also be obtained.

We consider now the finite open cover $\mathcal{E}_\gamma = \{B(x_i, \delta_1/8 + \gamma), i = 1, \dots, l\}$ of M and construct the partition $\mathcal{P}_\gamma$ induced by $\mathcal{E}_\gamma$ by the same procedure as before. Since $\gamma < \varepsilon < d_0$ we obtain $d(w, \partial B(x_i, \delta_1/8 + \gamma)) \geq d_0 - \gamma > 0$ for all $i = 1, \dots, l$, and every $w \in \mathcal{C}_0$. This shows that each $w \in \mathcal{C}_0$ is contained in some atom P_w of $\mathcal{P}_\gamma$. Moreover there cannot be distinct $w_1, w_2 \in \mathcal{C}_0$ such that $w_2 \in P_{w_1}$, because this would mean that for some $i \in \{1, \dots, l\}$ we have $w_2 \in B(x_i, \delta_1/8)$, $w_1 \notin B(x_i, \delta_1/8)$ and $w_1, w_2 \in B(x_i, \delta_1/8 + \gamma)$, which contradicts the choice of $\gamma < d_0$.

Let us consider $\{P_w, w \in \mathcal{C}_0\}$. There might be other (finitely many) atoms P in $\mathcal{P}_\gamma$ and, if so, we join them to some adjacent atom P_w, (meaning $\bar{P} \cap \bar{P}_w \neq \emptyset$) obtaining a new atom $P \cup P_w$. In this way we obtain a partition $\mathcal{P}_\varepsilon$ with as many atoms as the elements of $\mathcal{C}_0$ and satisfying items (1) and (2) of the statement of the lemma.

Clearly for any $w \in \mathcal{C}_0$ the corresponding atoms $P_w \in \mathcal{P}_\varepsilon$ and $Q_w \in \mathcal{P}$ satisfy

$$\mathrm{Leb}(P_w \Delta Q_w) \leq \sum_{i=1}^{l} \mathrm{Leb}\left(B\left(x_i, \frac{\delta_1}{8} + \gamma\right)\right) < l \cdot \frac{\varepsilon}{l} = \varepsilon$$

and $\mathrm{diam}(P_w) \leq 2(\delta_1/8 + \gamma) < \delta_1/2$. Since $\mathcal{P}$ is a finite partition with $\mathrm{Leb}(\partial\mathcal{P}) = 0$ we have $\iota = \min\{\mathrm{Leb}(P) : P \in \mathcal{P}\} > 0$ and so given $\delta > 0$ and taking $\varepsilon < \min\{\iota \cdot \delta, d_0\}$ we get

$$\mathrm{Leb}(P_w \Delta Qw) < \varepsilon = \iota \cdot \frac{\varepsilon}{\iota} < \iota \cdot \delta \leq \delta \cdot \mathrm{Leb}(q_w).$$

The proof is complete. □

Having this we can now obtain the following flexible covering lemma with hyperbolic preballs which will enable us to approximate the Lebesgue measure of a given set through the measure of families of hyperbolic preballs.

Lemma 3.5. *Let a measurable set* $E \subset M$, $m \geq 1$ *and* $\varepsilon > 0$ *be given with* $\mathrm{Leb}(E) > 0$. *Let* $\theta > 0$ *be a lower bound for the density of hyperbolic times for Lebesgue almost every point. Then there are integers* $m < n_1 < \cdots < n_k$ *for* $k = k(\varepsilon) \geq 1$ *and families* $\mathcal{E}_i$ *of subsets of* M, $i = 1, \dots k$ *such that*

(1) $\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_k$ *is a finite pairwise disjoint family of subsets of M;*
(2) n_i *is a* $(\sigma/2, \delta/2)$*-hyperbolic time for every point in P, for every element* $P \in \mathcal{E}_i$, $i = 1, \dots, k$;

(3) every $P \in \mathcal{E}_i$ is the preimage of some element $Q \in \mathcal{P}$ under an inverse branch of f^n, $i = 1, \ldots, k$;

(4) there is an open set $U_1 \supset E$ containing the elements of $\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_k$ with $\mathrm{Leb}(U_1 \backslash E) < \varepsilon$*;*

(5) $\mathrm{Leb}\,(E \Delta \cup_i \mathcal{E}_i) \leq (1 - \frac{\theta}{4})^k < \mathcal{E}$.

The proof follows (Ref. 43, Lemma 8.2) closely. We write $\mathcal{C}_m$ the set of pairs (z, n_i) where $f^{n_i}(z) = w \in \mathcal{C}_0$ and $z \in P$ for all $P \in \mathcal{E}i$ and $i = 1, \ldots, k$ (such z exist by item (3) of Lemma 3.5).

Remark 3.6 Note that k depends on ε only and not on the set E.

Proof: By the non-uniformly expanding assumption on f we know that there exists $\theta > 0$ such that Lebesgue almost every point has density $> \theta$ of hyperbolic times of f.

Let U_1 be an open set and K_1 a compact set such that $K_1 \subset E \subset U_1$ and $\mathrm{Leb}(U_1 \backslash K_1) < \varepsilon$ and $\mathrm{Leb}(K_1) > (1/2)\mathrm{Leb}(U_1)$. Using Lemma 3.3 with $B = K_1$ and $\nu = \mathrm{Leb}/\mathrm{Leb}(K_1)$ we can find $n_1 > m$ such that $e^{-cn_1} < d(K_1, M \backslash U_1)$ and the subset L_1 of points of K_1 for which n_1 is a hyperbolic time satisfies $\mathrm{Leb}(L_1) \geq \frac{\theta}{2}\mathrm{Leb}(K_1) \geq \frac{\theta}{4}\mathrm{Leb}(E)$.

Given $x \in L_1$ let $g : B(f^{n_1}(x), \delta_1) \to V_{n_1}(x)$ be the inverse branch of $f^{n_1} | V_{n_1}(x)$, recall that n_1 is a hyperbolic time for x and see Proposition 3.1. By the choice of $\mathcal{P}$ there exists a unique $P \in \mathcal{P}$ such that $f^{n_1}(x) \in P$. Let us consider $g(P)$ and let $\mathcal{E}_1$ be the family of all such sets obtained as $g(P)$ which intersect L_1, where g is an inverse branch of f^{n_1} corresponding to a hyperbolic time and P is an element of $\mathcal{P}$.

Note that the elements of $\mathcal{E}_1$ are pairwise disjoint because $\mathcal{P}$ is a partition. Moreover by the properties of hyperbolic times (Proposition 3.1) the diameter of $P \in \mathcal{E}_1$ is smaller than $e{-}^{cn_1}$. Hence the union E_1 of all the elements of $\mathcal{E}_1$ is contained in U_1 and by construction

$$\mathrm{Leb}(E_1 \cap E) \geq \mathrm{Leb}(L_1) \geq \frac{\theta}{4}\mathrm{Leb}(E).$$

Now consider the open set $U_2 = U_1 \backslash \overline{E_1}$ and set $K_2 \subset E \backslash \overline{E_1}$ a compact set such that $\mathrm{Leb}(K_2) \geq (1/2)\mathrm{Leb}(E \backslash E_1)$. Observe that $\mathrm{Leb}(\overline{E_1} \backslash E_1) = 0$ since $\partial \mathcal{P}$ has zero Lebesgue measure and this property is preserved under backward iteration by the regularity assumption on f. Reasoning as before, we can find $n_2 > n_1$ such that $e^{-cn_2} < d(K_2, M \backslash U_2)$ and a set $L_2 \subset K_2$ such that $\mathrm{Leb}(L_2) \geq \left(\frac{\theta}{2}\right) \mathrm{Leb}(K_2)$ and n_2 is a hyperbolic time for every $x \in L_2$. Let $\mathcal{E}_2$ be the family of elements $g(P)$ which intersect L_2, where $P \in \mathcal{P}$ and g is an inverse branch of f^{n_1} corresponding to a hyperbolic time.

Again $\mathcal{E}_2$ is a pairwise disjoint family of sets whose diameters are smaller than e^{-cn_2}. Thus their union E_2 is contained in U_2. Hence $\mathcal{E}_1 \cup \mathcal{E}_2$ is also a pairwise disjoint family and, in addition

$$\operatorname{Leb}(E_2 \cap (E \backslash E_1)) \geq \operatorname{Leb}(L_2) \geq \frac{\theta}{2}\operatorname{Leb}(K_2) \geq \frac{\theta}{4}\operatorname{Leb}(E \backslash E_1).$$

Repeating this procedure we obtain families $\mathcal{E}_i, i = 1, \dots k$ of elements of $\mathcal{P}_{n_i}$ which are pairwise disjoint and contained in U_1, and

$$\operatorname{Leb}\left(E_{i+1} \cap (E \backslash (E_1 \cup \cdots \cup E_i))\right) \geq \frac{\theta}{4}\operatorname{Leb}(E \backslash (E_1 \cup \cdots \cup E_i)) \tag{14}$$

for all $i = 1, \dots, k-1$, for some $k \geq 1$, where $E_j = \cup \mathcal{E}_j$. Hence

$$\operatorname{Leb}\left(\bigcup_{i=1}^{k} E_i \backslash E\right) \leq \operatorname{Leb}(U_1 \backslash E) < \varepsilon$$

and (14) ensures that

$$\operatorname{Leb}\left(E \Big\backslash \bigcup_{i=1}^{k} E_i\right) \leq \left(1 - \frac{\theta}{4}\right)^k \operatorname{Leb}(E).$$

Therefore we can find $k \geq 1$ such that $\operatorname{Leb}(E \,\Delta \cup_{i=1}^{k} \mathcal{E}_i) < \varepsilon$, as stated. □

Remark 3.7. Note that the construction proving Lemma 3.5 gives a finite sequence of hyperbolic times, open sets $U_1, \dots, U_k$ and closed sets $\bar{E}_1, \dots, \bar{E}_k$. Having these we can find small enough $\delta > \varepsilon > 0$, replace $\mathcal{P}$ in the proof of Lemma 3.5 by any partition $\mathcal{P}_\varepsilon$ obtained as in Lemma 3.4 (by slightly modifying $\mathcal{P}$), and use the same inverse branches to obtain families $\mathcal{E}_i'$ of preballs such that

$$\operatorname{Leb}\left(\left(\bigcup_i \mathcal{E}_i\right) \Delta \left(\bigcup_i \mathcal{E}_i'\right)\right) \leq \sum_i C_1 \delta \operatorname{Leb}(\mathcal{E}_i) < C_1 \delta \operatorname{Leb}\left(\bigcup_i \mathcal{E}_i\right) \leq C_1 \delta$$

where C_1 is the volume distortion constant (see Proposition 3.1). Hence after the modification of the initial partition we get

$$\operatorname{Leb}\left(E \Delta \bigcup_i \mathcal{E}_i'\right) < \varepsilon + C_1 \delta < (1 + C_1)\delta$$

since $\varepsilon < \delta$. Moreover the set $\mathcal{C}_m$ is unaffected since $\mathcal{C}_0$ is fixed and the inverse branches are kept.

3.2. The Partially Hyperbolic Setting

Here we state the main results needed to obtain an extension of the covering Lemma 3.5 to the setting of partially hyperbolic non-uniformly expanding attracting sets. As we indicate along the way, the proofs of most of them can be found in Ref. 6.

3.2.1. Stable/Unstable Cone Fields

Let Λ be a partially hyperbolic and nonuniformly expanding attracting set for a C^2 diffeomorphism $f : M \to M$ with a trapping region $U \subset M$. The existence of the dominated splitting $E \oplus F$ of $T_\Lambda M$ ensures the existence of a continuous extension $\tilde{E} \oplus \tilde{F}$ of $E \oplus F$ to a neighborhood of Λ, which we assume without loss to be U, and of the following cone fields:

$$\textbf{stable cones}: \mathbb{E}_x^a = \{(u, v) \in \tilde{E}(x) \oplus \tilde{F}(x) : \|v\| \leq a \cdot \|u\|\};$$
$$\textbf{unstable cones}: \mathbb{F}_x^b = \{(u, v) \in \tilde{E}(x) \oplus \tilde{F}(x) : \|u\| \leq b \cdot \|v\|\};$$

for all $x \in U$ and $a, b \in (0, 1)$, which are Df-invariant in the following sense (see e.g. Ref. 17, Appendix C)

- if $x, f^{-1}(x) \in U$, then $Df^{-1}(\mathbb{E}_x^a) \subset \mathbb{E}_{f^{-1}(x)}^{\lambda a}$;
- if $x, f(x) \in U$, then $Df(\mathbb{F}_x^b) \subset \mathbb{F}_{f(x)}^{\lambda b}$;

for some $\lambda \in (0, 1)$. Continuity enables us to unambiguously denote $d_E = \dim(\tilde{E})$ and $d_F = \dim(\tilde{F})$, so that $d = d_E + d_F = \dim(M)$, and domination guarantees that the angles between the $\tilde{E}$ and $\tilde{F}$ directions are bounded from below away from zero at every point.

3.2.2. Hyperbolic Times

In this setting, given $\sigma > 1$ we say that n is a σ-hyperbolic time for $x \in U$ if

$$\prod_{j=n-k+1}^{n} \|(Df|F_{f^j(x)})^{-1}\| \leq \sigma^k \quad \text{for all} \quad 1 \leq k \leq n.$$

Remark 3.8. This definition of hyperbolic time is entirely analogous to the one given in the local diffeomorphisms setting except that we restrict the derivatives to the F-direction. Hence the statement and proof of Lemma 3.3 carry over without change.

3.2.3. E-Disks and F-Disks

Let us fix the unit balls of dimensions d_E, d_F

$$\mathbb{B}_E = \{w \in \mathbb{R}^{d_E} : \|w\|_2 \le 1\} \quad \text{and} \quad \mathbb{B}_F = \{w \in \mathbb{R}^{d_F} : \|w\|_2 \le 1\}$$

where $\|\cdot\|_2$ is the standard Euclidean norm on the corresponding Euclidean space. We say that a $C^{1+\alpha}$ embedding $\Delta : \mathbb{B}_E \to M$ (respectively $\Delta : \mathbb{B}_F \to M$) is a *$E$-disk* (resp. *$F$-disk*) if the image of $D\Delta(w)$ is contained in $\mathbb{E}^a_{\Delta(w)}$ for all $w \in \mathbb{B}_E$ (resp. $D\Delta(w)(\mathbb{R}^{d_F}) \subset \mathbb{F}^b_{\Delta(w)}$ for every $w \in \mathbb{B}_F$), where $\alpha \in (0, 1)$ if fixed.

3.2.4. Curvature of E- and F-Disks at Hyperbolic Times

Let $r_0 > 0$ be an injectivity radius of the exponential map on M, that is $\exp_x : B(x, r_0) \to M$ is a diffeomorphism onto its image $G(x, r_0) = \exp_x(B(x, r_0))$, where $B(x, r_0) = \{v \in T_X M : \|v\| < r_0\}$ is the r_0-neighborhood of 0 in $T_x M$. By the continuity of the splitting $E \oplus F$ and the cone fields we can choose $0 < r < \min\{r_0, \delta_1/4\}$ such that for every $x \in \Lambda$ the subspace E_x is contained in all the images of the cone field $\mathbb{E}^a_x$ under the exponential map $\exp_x$ and analogously for the complementary direction, that is for every $y \in G(x, r) \cap \Lambda$ we have

$$E_x \subset D\big(\exp_x^{-1}\big)(\mathbb{E}^a_y) \quad \text{and} \quad F_x \subset D\big(\exp_x^{-1}\big)(\mathbb{F}^b_y). \tag{15}$$

This ensures that every F-disk (respectively every E-disk) Δ is such that its image on $B(x, r)$ given by $\exp_x^{-1}(\Delta \cap G(x, r))$ is transversal to the direction of E_x (resp. F_x).

The "curvature" of E- and F-disks can be determined by the notion of Hölder variation of the tangent bundle as follows.

We write Δ also for the image of the respective embedding for every E- or F- disk. Hence if Δ is a E-disk and $y = \Delta(w)$ for some $w \in \mathbb{B}_E$, then the tangent space of Δ at y is the graph of a linear map $A_x(y) : T_x\Delta \to F(x)$ for $w \in \Delta^{-1}(V_x)$ (here $T_x\Delta = D\Delta(x)(\mathbb{R}^{d_E})$). The same happens locally for a F-disk exchanging the roles of the bundles E and F above.

The domination condition on the splitting $E \oplus F$ ensures the existence of $\zeta \in (0, 1)$ such that for some $n \ge 1$ and all $x \in \Lambda$

$$\|Df^n|E_x\| \cdot \|(Df^n|F_x)^{-1}\|^{1+\zeta} \ge \frac{3}{4}.$$

Given $C > 0$ we say that the *tangent bundle of* Δ is (C, ζ) -Hölder if

$$\|A_x(y)\| \ge C\mathrm{dist}_\Delta(x, y)^\zeta \quad \text{for all} \quad y \in G(x, r) \cap \Delta \quad \text{and} \quad x \in U, \tag{16}$$

where $\mathrm{dist}_\Delta(x, y)$ is *the distance along* Δ defined by the length of the shortest smooth curve from x to y inside Δ calculated with respect to the Riemannian norm $\|\cdot\|$ induced on *TM*.

For a E- or F-disk $\Delta \subset U$ we define

$$\kappa(\Delta) = \inf\{C > 0 : T\Delta \quad \text{is} \quad (C, \zeta)\text{-Hölder}\}. \tag{17}$$

The proof of the following result can be found in (Ref. 6, Sec. 2.1). The basic ingredients are the cone invariance and dominated decomposition properties for f.

Proposition 3.9. *There is $C_2 > 0$ such that given a F-disk $\Delta \subset U$*

(1) there exists $n_1 \in \mathbb{N}$ such that $\kappa(f^n(\Delta)) \leq C_2$ for all $n \geq n_1$;
(2) if $\kappa(\Delta) \leq C_2$ then $\kappa(f^n(\Delta)) \leq C_2$ for all $n \geq 0$;
(3) in particular, if Δ is as in the previous item, then

$$J_n : f^n(\Delta) \ni x \mapsto \log|\det(Df|T_x(f^n(\Delta))|$$

is (L_1, ζ)-Hölder continuous with $L_1 > 0$ depending only on C_2 and f, for every $n \geq 1$.

3.2.5. Distortion Bounds

The following uniform backward contraction and distortion bounds are proved in (Ref. 6, Lemma 2.7, Proposition 2.8).

Proposition 3.10. *There exist $C_3, \delta_1 > 0$ depending only on f, σ such that, given any F-disk $\Delta \subset U$, $x \in \Delta$, and $n \geq 1_D$ a σ-hyperbolic time for x,*

(1) $\text{dist}_{f^{n-k}(D)}(f^{n-k}(f\frac{n}{(y)}), f^{n-k}(x)) \leq \sigma^{k/2}\,\text{dist}\,(f^n(D)f^n(y)), (f^n(x))$, *for all $y \in \Delta$ with* $\text{dist}(f^n(x), f^n(y)) \leq \delta_1$*;*
(2) if $\kappa(\Delta) \geq C_2$ then

$$\frac{1}{C_3} \leq \frac{|\det Df^n|T_y\Delta|}{|\det Df^n|T_x\Delta|} \geq C_3$$

for every $y \in \Delta$ such that $\text{dist}(f^n(y), \quad f^n(x)) \geq \delta_1$.

3.2.6. The Initial Partition and the Covering Lemma

Now we consider the following rectangle

$$\hat{R}(x, s) = \{(u, v) \in T_xM : \|u\| < s, \|v\| < s, u \in E_x, v \in F_x\}$$

where s is chosen so that $\hat{R}(x, s) \subset B_x(r)$ for all $x \in \Lambda$. This defines an open cover $\{\exp_x(\hat{R}(x, s))\}_{x\in\Lambda}$ of Λ which admits a finite subcover denoted by $\mathcal{R} = \{R_1 = R(x_1, s), \ldots, R_h = R(x_h, s)\}$. This finite cover will define the initial partition $\mathcal{P}$ given by

$$\mathcal{P} = \{R_1, M\backslash R_1\} \vee \cdots \vee \{R_h, M\backslash R_h\}.$$

We may assume without loss that $\mathrm{Leb}(\partial\mathcal{P}) = 0$ by slightly changing the initial cover. We choose an interior point in each element of $\mathcal{P}$ which together define the set $\mathcal{C}_0$.

Now we adapt the covering Lemma 3.5 to the setting of partially hyperbolic non-uniformly expanding attracting sets as follows.

Lemma 3.11. *Let a measurable set $E \subset U$, $m \geq 1$ and $\varepsilon > 0$ be given. Let $\theta > 0$ be a lower bound for the density of hyperbolic times for Lebesgue almost every point on U. Then there are integers $m < n_1 < \cdots < n_k$ for $k = k(\varepsilon) \geq 1$, and families $\mathcal{E}_i$ of subsets of M, $i = 1, \ldots, k$ such that*

(1) $\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_k$ *is a finite family of subsets of M and each $\mathcal{E}_i$ is a pairwise disjoint family;*
(2) n_i *is a $(\sigma/2, \delta/2)$-hyperbolic time for every point in P, for every element $P \in \mathcal{E}_i, \quad i = 1, \ldots, k$;*
(3) *every $P \in \mathcal{E}_i$ is the preimage of some element $Q \in \mathcal{P}$ under $f^{-n_i}, \quad i = 1, \ldots, k$;*
(4) $\mathrm{Leb}\,(E \setminus \cup_i \mathcal{E}_i) \leq (1 - \frac{\theta}{4})^k < \varepsilon$.

Proof: Let $E \subset U$, $\varepsilon > 0$ and $m \geq 1$ be given. Set $\nu = \mathrm{Leb}/\mathrm{Leb}(E)$ and apply Lemma 3.3 with $B = E$ to obtain $n_1 > m$ and $L_1 \subset E$ such that n_1 is a hyperbolic time for every point $x \in L_1$ and $\mathrm{Leb}(L_1) \geq \frac{\theta}{2}\mathrm{Leb}(E)$.

Given $x \in L_1$ let P_x be the unique element of the partition $f^{-n_1}\mathcal{P}$ which contains x (recall that f is a diffeomorphism). Define $\mathcal{E}_1 = \{P_x : x \in L_1\}$. Then $\mathcal{E}_1$ is a finite pairwise disjoint family of preimages of elements of $\mathcal{P}$ corresponding to a hyperbolic time n_1. If E_1 is the union of the elements of $\mathcal{E}_1$, then

$$\mathrm{Leb}(E_1 \cap E) \geq \mathrm{Leb}(L_1) \geq \frac{\theta}{2}\mathrm{Leb}(E).$$

Now consider $\hat{E}_2 = E \setminus \overline{E}_1$. If $\mathrm{Leb}(\hat{E}_2) < \varepsilon$ then we are done, since then $\mathrm{Leb}(E \setminus E_1) < \varepsilon$ because $\mathrm{Leb}(\partial\mathcal{E}_1) = 0$ as f is regular map. Otherwise use again Lemma 3.3 to find $n_2 > n_1$ and $L_2 \subset \hat{E}_2$ such that n_2 is a hyperbolic time for all points of L_2 and $\mathrm{Leb}(L_2) \geq \frac{\theta}{2}\mathrm{Leb}(\hat{E}_2)$.

Let $\mathcal{E}_2$ be the family of all elements of the partition $f^{-n_2}\mathcal{P}$ which intersect $\hat{E}_2$. Then $\mathcal{E}_2$ is a pairwise disjoint family and the union E_2 of its elements satisfies

$$\mathrm{Leb}(E_2 \cap (E \setminus E_1)) \geq \mathrm{Leb}(L_2) \geq \frac{\theta}{2}\mathrm{Leb}(\hat{E}_2) \geq \frac{\theta}{4}\mathrm{Leb}(E \setminus E_1).$$

Repeating this procedure we get families $\mathcal{E}_i, i = 1, \ldots, k$ of elements of $f^{-n_i}\mathcal{P}$ with $m < n_1 < \cdots < n_k$ satisfying the inequality (14). These families satisfy items (1)–(3) by construction and item (4) follows by (14) as in the proof of Lemma 3.5. This concludes the proof. □

Observe that we may apply Lemma 3.4 to $\mathcal{P}$ to ensure that, for a given denumerable family of f-invariant probability measures, there is a partition $\mathcal{P}_\varepsilon$ arbitrarily close to $\mathcal{P}$, with the same number of elements, such that the measure of the boundary of the elements of $\mathcal{P}_\varepsilon$ is zero with respect to all measures of the family. Moreover as in the previous subsection, we write $\mathcal{C}_m$ the set of pairs (z, n_i) where $f^{n_i}(z) = w \in \mathcal{C}_0$ and $z \in P$ for all $P \in \mathcal{E}_i$ and $i = 1, \ldots, k$. In addition, we can build the new partition $\mathcal{P}_\varepsilon$ in such a way that the sets $\mathcal{C}_n$ are unchanged.

3.3. The Volume of Dynamical Balls

Here we show that the volume of dynamical balls on hyperbolic times is well controlled by $S_n J$, either in the local diffeomorphism case with or without singularities, or in the partially hyperbolic case.

3.3.1. *The Local Diffeomorphism Case with Singularities*

Note that by the properties of bounded distortion of volumes during hyperbolic times (item 3 of Proposition 3.1) we can write, if n is a hyperbolic time of f for $x \in M$

$$\begin{aligned}\mathrm{Leb}(B(f^k(x), n-k, \delta_1)) &= \int_{B(f^k(x),n-k,\delta_1)} \frac{dz}{|\det Df^{n-k}(z)|} \\ &\le C_1 \frac{\mathrm{Leb}(B(f^n(x), \delta_1))}{|\det Df^{n-k}(x)|},\end{aligned}$$

then recalling that $J = \log|\det Df|$ we get

$$\begin{aligned}\mathrm{Leb}(B(f^k(x, n-k, \delta_1)) &\le C_1 e^{-S_{n-k}J(f^k(x))}\mathrm{Leb}(B(f^n(x), \delta_1) \\ &\le C_1 e^{-S_{n-k}J(f^k(x))}.\end{aligned}$$

Observe that by Proposition 3.1 if n is a hyperbolic time of f for x we get due to uniform backward contraction

$$S_{n-k}J(f^k(x)) = \log|\det Df^{n-k}(x)| \ge (n-k)\cdot \dim(M)\log\sigma/2 > 0$$

which will be used several times in what follows.

3.3.2. *The Partially Hyperbolic Case with Non-Uniform Expansion*

In the partially hyperbolic and non-uniformly expanding setting we recall the construction of the cover $\mathcal{R} = \{R_1, \ldots, R_j\}$ and the initial partition $\mathcal{P}$ from Sec. 3.2. Observe that if we take δ_0 to be the Lebesgue number of the covering $\mathcal{R}$ (see e.g. Ref. 42), then for all $0 < \delta < \delta_0$ we have for all $x \in U$ and $n \ge 1$ a hyperbolic time for x

$$B(x, n, \delta) \subset f^{-n}\mathcal{P}(x),$$

where $f^{-n}\mathcal{P}(x)$ denotes the element of $f^{-n}\mathcal{P}$ which contains x. To find an upper bound for the volume of this dynamical ball it is enough to estimate the volume of $f^{-n}\mathcal{P}(x)$ when n is a hyperbolic time for x.

Let $P \in \mathcal{P}$ be such that $f^{-n}(P)$ has a positive Lebesgue measure subset $\tilde{P}$ of points for which n is a hyperbolic time and choose h such that $R_h \supset P$. Let $\tilde{Q} \in \mathcal{P}$ be such that $Q = \tilde{Q} \cap \tilde{P}$ has positive Lebesgue measure and choose l such that $R_l \supset Q$.

We consider the projection of $\tilde{P} = \exp_{x_l}^{-1}(\tilde{P})$ on E_{x_l} parallel to F_{x_l}. Its diameter will be bounded by a constant which is a function of f and s only, since the number of different R_l is finite. Projecting $\tilde{Q}$ on the complementary direction F_{x_l} parallel to E_{x_l} we may use the backward contraction and bounded area distortion for hyperbolic times along F-disks to estimate the area along F-disks and integrate to deduce a volume estimate.

Indeed, observe that since the E direction is uniformly contracted by Df, if we fix a point $x_0 \in Q$, the corresponding point $x_n = f^n(x_0) \in P \cap f^n(Q)$ and a E-disk γ which crosses R_h, then the connected component $\tilde{\gamma}$ of $f^{-n}(\gamma) \cap R_l$ containing x_0 is a E-disk which also crosses R_l. Moreover distances along γ are uniformly expanded by f^{-1}. Thus every point $w_0 \in \tilde{\gamma}$ is such that $w_k = f^k(w_0)$ and $x_k = f^k(x_0)$ satisfy

$$C\frac{\delta_1}{4} > C_S \geq \text{dist}(w_0, x_0) \geq C\lambda^{-k}\text{dist}(w_k, x_k), \tag{18}$$

for some constant $C > 0$ depending on f only. Hence if we take s small enough then we can ensure that w_k is close enough to x_k for $k = 1, \ldots, n$ so that n is also an hyperbolic time for all $w_0 \in \tilde{\gamma}$. Thus we can consider F-disks β_q through the points q of Q parallel to F, which are transversal to $\tilde{\gamma}$. Then the images $f^n(\beta_q)$ will be F-disks crossing R_l which together cover $P \cap f^n(Q)$, see Figure 1.

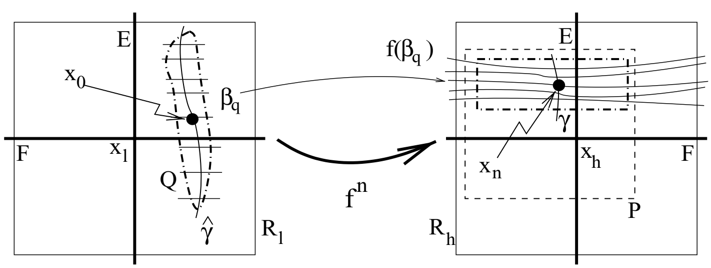

Fig. 1. The diameter of the elements of $\mathcal{E}_n$ through the use of E-disks and images of F-disks on a hyperbolic time.

The preimages $f^{-n}(P \cap f^n(Q) \cap f^n(\beta_n))$ then form a cover of Q and these predisks are F-disks whose diameter is smaller than e^{-cn}.

Using Tonelli's Theorem we can write $\text{Leb}(Q) = \int_{\hat{\gamma}} m(Q \cap \beta_q) dq$ where m denotes the d_F-dimensional Lebesgue measure induced by Leb on F-disks and dq is Lebesgue measure along the disk $\hat{\gamma}$. By the Change of Variables Formula together with the bounded area distortion along hyperbolic times in the partially hyperbolic setting given by Proposition 3.10 we get for each $q \in \hat{\gamma}$

$$\begin{aligned} m(Q \cap \beta_q) &= \int_{\beta_q} \chi_Q dm = \int_{f^{-n}(f^n(\beta_q))} \chi_Q dm \\ &= \int_{f^n(\beta_q)} (\chi_Q \circ f^{-n}) \cdot |\det Df^{-n}| f^n(\beta_q)| dm \\ &= \int_{f^n(\beta_q)} e^{-S_n J(f^{-n}(z))} \chi_{f^n(Q)}(z)\, dm(z) \\ &\leq C_3 \cdot e^{-S_n J(f^{-n}(q))} \cdot m(f^n(Q) \cap f^n(\beta_q)), \end{aligned}$$

thus $\text{Leb}(Q) \leq \int_{\hat{\gamma}} C_3 e^{-S_n J(q)} m(f^n(Q) \cap f^n(\beta_q)) dq$. But by (18) we see that every $q \in \hat{\gamma} \cap Q$ satisfies

$$d(f^k(q), f^k(x)) \leq C\lambda^k \frac{\delta_1}{4}, \quad \text{for} \quad k = 0, \ldots, n.$$

Hence because J is at least $C^{1+\alpha}$ for some $\alpha \in (0, 1)$ with Hölder constant $C > 0$ (in fact we can take $\alpha = 1$ if f is C^2) the usual bounded distortion argument provides a constant $C_0 > 0$ such that

$$\log \frac{|\det Df^n | F_q|}{|\det Df^n | F_x|} = \sum_{j=0}^{n-1} \log \frac{|\det Df(f^j(q))|}{|\det Df(f^j(x))|} \leq \sum_{j=0}^{n-1} C d(f^j(q), f^j(x))^\alpha \leq C_0.$$

Hence $|S_n J(q) - S_n J(x)| \leq C_0$ and by the above integration estimates we get

$$\text{Leb}(Q) \leq \int_{\hat{\gamma}} C_3 e^{C_0} e^{-S_n J(x)} m(f^n(Q) \cap f^n(\beta_q)) dq \leq \tilde{C}\, e^{-S_n J(x)},$$

where $\tilde{C}$ is bounded by the d_E-dimensional area A_E of $\hat{\gamma}$ (which is a function of $s < \delta_1/4$) times a uniform bound A_F for the d_F-dimensional area of $f^n(\beta_q)$ (which is a function of the curvature bound C_2 from Proposition 3.9 and of δ_1, see Fig. 1) multiplied by the bounded distortion constants, that is $\tilde{C} \leq C_3 e^{C_0} A_E A_F$.

This shows that we have the same kind of estimate for the volume of a dynamical ball as in the local diffeomorphism case, except for a different distortion constant and the fact that the Jacobian is calculated along the F direction.

4. HYPERBOLIC TIMES AND LARGE DEVIATIONS

The statements of the main theorems and corollaries are consequences of the following more abstract result.

Theorem 4.1. *Let $f : M \to M$ be a local diffeomorphism outside a non-flat singular set $\mathcal{S}$ admitting $\sigma \in (0, 1)$ and $b, \delta > 0$ such that Lebesgue almost every point has positive density of (σ, δ, b)-hyperbolic times. Then given $c \in \mathbb{R}$ and a continuous function $\varphi : M \to \mathbb{R}$ items (1)–(3) of Theorem A hold.*

Clearly Theorem A follows from Theorem 3.2 together with Theorem 4.1. Moreover item (1) in the statement of Theorem A is just item (1) of (Ref. 60, Theorem 1) so it will not be proved here.

4.1. Upper Bound for Large Deviations

Here we prove the upper bound in item 2 of Theorem 4.1.

Let $\varphi : M \to \mathbb{R}$ be a fixed continuous function. Consider for $n \geq 1$ and some fixed $\varepsilon, \delta, c > 0$

$$A_n = A_n(\delta, \varepsilon) = \left\{ x : \frac{1}{n} S_n \Delta_\delta(x) \leq \varepsilon \right\} \quad \text{and} \quad B_n = \left\{ x : \frac{1}{n} S_n \varphi(x) \geq c \right\}.$$

Since we want to bound a limit superior from above, we can assume without loss that $\text{Leb}(A_n \cap B_n) > 0$ in what follows. We fix a partition $\mathcal{P}$ of M as in Sec. 3.1 (whose diameter is smaller than $\delta_1/4$) and use Lemma 3.5 with $m = n$, $E \subset U_1 \subset A_n \cap B_n$ such that U_1 is open and

$$\text{Leb}\,((B_n \cap A_n) \setminus E) < \frac{1}{2n} \text{Leb}\,(B_n \cap A_n),$$

which can be done since $S_n\varphi$ is continuous and $S_n \Delta_\delta$ is upper- semicontinuous. Then we can find a family $\mathcal{U}_n = \mathcal{E}_i \cup \cdots \cup \mathcal{E}_k$ of hyperbolic preballs contained in U_1 satisfying

$$\text{Leb}(E \Delta \bigcup \mathcal{U}_n) \leq \left(1 - \frac{\theta}{4}\right)^k < \frac{1}{2n} \text{Leb}(A_n \cap B_n).$$

Note that $\text{Leb}((A_n \cap B_n) \setminus \mathcal{U}_n) \leq \text{Leb}((A_n \cap B_n) \setminus E) + \text{Leb}(E \setminus \mathcal{U}_n) < \frac{1}{n} \text{Leb}(A_n \cap B_n)$ and so

$$\text{Leb}(A_n \cap B_n) < \frac{n}{n-1} \text{Leb}(\mathcal{U}_n). \tag{19}$$

Observe also that for any element $P \in \mathcal{E}_i$ there exists $x \in M$ and a hyperbolic time h_i of f for x such that $P \subset B(x, h_i, \delta_1)$, by construction, where $i = 1, \ldots, k_n$

and $n < h_1 < \cdots < h_{k_n}$. Let $\mathcal{C}_n$ be the set of all such pairs (x, h_i), one for each element of $\mathcal{U}_n$ and to simplify the notation we write h_n for h_{k_n}.

Following the arguments in the proof of (Ref. 60, Theorem 1(2)) we consider the measure

$$\sigma_n = \frac{1}{Z_n} \sum_{(x,l)\in\mathcal{C}_n} e^{-S_l J(x)} \cdot \delta_x \quad \text{where} \quad Z_n = \sum_{(x,l)\in\mathcal{C}_n} e^{-S_l J(x)}.$$

Note that by definition each element of the partition $\bigvee_{i=0}^{h_n-1} f^{-i}\mathcal{P}$ contains at most the first coordinate of one element of $\mathcal{C}_n$. Thus using (Ref. 59, Lemma 9.9) we have

$$H\sigma_n\left(\bigvee_{i=0}^{h_{n-1}} f^{-1}\mathcal{P}\right) - \int S_{l(x)}J(x)d\sigma_n(x) = \log \sum_{(x,l)\in\mathcal{C}_n} e^{-S_l J(x)},$$

where we write $l(x)$ for the unique integer l such that $(x, l) \in \mathcal{C}_n$. Since $S_{l(x)-n}J(f^n(x)) > 0$ (see Sec. 3.3) and $l(x) > n$ we get

$$H\sigma_n\left(\bigvee_{i=0}^{h_{n-1}} f^{-i}\mathcal{P}\right) - \int S_n J d\sigma_n \geq \log \sum_{(x,l)\in\mathcal{C}_n} e^{-S_l J(x)}. \tag{20}$$

Setting $\mu_n = \frac{1}{n}\sum_{i=0}^{n} f_*^i\sigma_n$ and μ a weak* accumulation point of μ_n, we may modify the initial partition $\mathcal{P}$ according to Lemma 3.4 and Remark 3.7 so that its diameter is smaller than $\delta_1/2$ and $\mu(\partial\mathcal{P}) = 0$ without loss, keeping $\mathcal{C}_n$ unchanged. As in (Ref. 59, p. 220) from the above we can deduce that for every $q \geq 1$

$$\limsup_{n\to+\infty} \frac{1}{n}\log Z_n \leq \frac{1}{q}\limsup_{n\to+\infty} H_{\mu_n}\left(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P}\right) + \lim\sup_{n\to+\infty}\int -J d\mu_n \tag{21}$$

$$\leq h_\mu(f, \mathcal{P}) - \int J d\mu \leq h_\mu(f) - \int J d\mu \tag{22}$$

if f is a local diffeomorphism, ensuring that μ, is f-invariant and that J is a continuous function (in this case $\mathcal{S} = \emptyset$ and Δ_δ plays no role, we may take $\Delta_\delta \equiv 0$ and $A_n = M$). Observe that since the points in $\mathcal{C}_n$ are contained in B_n and μ_n is a linear convex combination of measures of the form $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}$, we get for all $n \geq 1$

$$\int \varphi\mu_n = \frac{1}{n}\sum_{j=0}^{n-1}\sigma_n(\varphi\circ f^j) = \frac{1}{Z_n}\sum_{(x,l)\in\mathcal{C}_n} e^{-S_j J(x)} \cdot \frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x))$$

$$\geq c\cdot\frac{1}{Z_n}\sum_{(x,l)\in\mathcal{C}_n} e^{-S_l J(x)} = c \tag{23}$$

and hence $\int \varphi d\mu \geq c$ also because φ is a continuous function.

Note that from (19) and by Sec 3.3 we get for some constant $C > 0$

$$\begin{aligned}\text{Leb}(B_n) &\leq \frac{n}{n-1}\text{Leb}(\mathcal{U}_n) \leq \frac{n}{n-1}\sum_{(x,l)\in\mathcal{C}_n}\text{Leb}\left(B(x,l,\delta_1)\right)\\ &\leq \frac{n}{n-1}\sum_{(x,l)\in\mathcal{C}_n} Ce^{-S_l J(x)} = \frac{Cn}{n-1}Z_n. \end{aligned} \tag{24}$$

Therefore we have shown that there exists $\mu \in \mathcal{M}_f$ such that $\int \varphi\, d\mu \geq c$ and

$$\limsup_{n\to+\infty}\frac{1}{n}\log\text{Leb}(B_n) \leq \limsup_{n\to+\infty}\frac{1}{n}\log Z_n \leq h_\mu(f) - \int J d\mu,$$

which completes the proof of item 2 in the statement of Theorem 4.1 and Theorem A.

4.2. Upper Bound for Partially Hyperbolic Diffeomorphisms

Here we show that a bound similar to the one in item 2 of Theorem A also holds in the case of a partially hyperbolic non-uniformly expanding attracting set.

Let $f : M \to M$ be a diffeomorphism satisfying the conditions of Theorem D, let $\varphi : M \to \mathbb{R}$ be a continuous function, fix a real number c and set $J = \log|\det Df|f|$. Observe that since we have Lemma 3.11 we may argue exactly as in the previous subsection to arrive at an inequality just like (20).

Again as in the previous subsection we consider $\mu_n = \frac{1}{n}\sum_{i=0}^{n} f_*^i\sigma_n$ and μ a weak* accumulation point of μ_n. We also modify the partition $\mathcal{P}$ in such a way that the boundaries of each atom have zero measure with respect to all measures μ and μ_n, $n \geq 1$.

The inequality (20) enables us to obtain inequalities (21) and (22) exactly as before. Together with the volume estimates obtained in Sec. 3.3.2 we can then arrive also at inequality (24) just by using a different distortion constant and replacing the Jacobian of f by the Jacobian of f *along the F direction*. Hence we obtain the upper bound given by item 2 of Theorem A also in the setting of partially hyperbolic non-uniformly expanding attracting sets. This will be very useful to deduce Theorem D in Sec. 5.1.

4.3. Upper Bound with Singular/Critical Set

To obtain an analogous result to (22) in the limit with a transformation f with non-flat singularities, thus proving item 3 from Theorem A and Theorem 4.1, we need some extra work. Note that the same arguments lead us to (21) as before and, since the points in $\mathcal{C}_n$ are contained in $A_n \cap B_n$, by the same calculations (23) above we also get $\int \Delta_\delta d\mu_n \leq \varepsilon$ for every $n \geq 1$.

Lemma 4.2. *The singular set $\mathcal{S}$ has null μ measure.*

Proof: Arguing by contradiction, assume that $\mu(\mathcal{S}) > 0$. Then there exists $a > 0$ such that $\mu(B(\mathcal{S}, \eta)) \geq a$ for all $\eta > 0$. Let $\eta > 0$ be chosen so that $\mu(\partial B(\mathcal{S}, \eta)) = 0$ and $\inf_{B(\mathcal{S},\eta)} \Delta_\delta \geq 4\varepsilon/a$.

On the one hand, since μ is a weak* limit point of μ_n, there exists n_0 such that for $n > n_0$ we have $\mu_n(B(\mathcal{S}, \eta)) \geq a/2$. On the other hand, since $\Delta_\delta \geq 0$ we get by the choice of η

$$\frac{4\varepsilon}{a}\mu_n(B(\mathcal{S}, \eta)) \leq \mu_n(\Delta_\delta \cdot \chi_{B(\mathcal{S},\eta)}) \leq \mu_n(\Delta_\delta) \leq \varepsilon,$$

where $\chi_{B(\mathcal{S},\eta)}$ is the characteristic function of $B(\mathcal{S}, \eta)$, from which we deduce that $\mu_n(B(\mathcal{S}, \eta)) \leq a/4$. This contradiction shows that $\mu(\mathcal{S}) = 0$ and concludes the proof. $\square$

Lemma 4.3. *The functions Δ_δ, J and ψ are μ-integrable.*

Proof: Let us define the sequence of functions

$$\Delta_\delta^k = \xi_k \circ \Delta_\delta \text{ where } \xi_k(x) = \begin{cases} k & \text{if } |x| \geq k \\ x & \text{if } |x| < k \end{cases}, \qquad k \geq 1.$$

For $k > k_0$ with $k_0 > |\log(\delta/2)|$ and fixing $\eta > 0$, since Δ_δ^k is continuous and $\Delta_\delta \geq \Delta_\delta^k$ there is an integer n_0 such that for all $n > n_0$ we have

$$\mu\big(\Delta_\delta^k\big) \leq \mu_n\big(\Delta_\delta^k\big) + \eta \leq \mu_n(\Delta_\delta) + \eta \leq \varepsilon + \eta.$$

Since this holds for all $k \geq k_0$ and $\Delta_\delta(x) \to \infty$ when $x \to \mathcal{S}$, we have proved

$$\int_{M\setminus\mathcal{S}} \Delta_\delta d\mu < \infty.$$

Thus we get $\Delta_\delta \in L^1(\mu)$ since $\mu(\mathcal{S}) = 0$ by Lemma 4.2.

For the other functions, note that by conditions (S2) and (S3) on the singular set $\mathcal{S}$ we see that there exists a constant $\zeta > \beta$ such that on a small neighborhood V of $\mathcal{S}$ we have

$$|\log \|Df(x)^{-1}\|| + |\log|\det Df(x)^{-1}|| \leq \zeta\, |\log d(x, \mathcal{S})| \tag{25}$$

and since f is a local diffeomorphism on $M\setminus\mathcal{S}$, the μ-integrability of Δ_δ implies that of ψ and J. This concludes the proof of the lemma. $\square$

Lemma 4.4. *The measure μ is f-invariant.*

Proof: Since by Lemma 4.2 $\mu(\mathcal{S}) = 0$ we can find a sequence $\eta_n \to 0$ of positive numbers such that $\mu(\partial B(\mathcal{S}, \eta_n)) = 0$ for all $n \geq 1$ and $\mu(B(\mathcal{S}, \eta_n)) \to 0$ when $n \to \delta$.

Let us fix $\eta > 0$ and a continuous function $h : M \to \mathbb{R}$. Take n_0 such that

$$\mu(B(\mathcal{S}, \eta_n)) \cdot \sup |h| < \frac{\eta}{2}$$

for all $n > n_0$ and fix $n_1 > n_0$ such that

$$\frac{1}{2}\mu(B(\mathcal{S}, \eta_n)) \leq \mu_n(B(\mathcal{S}, \eta_n)) \leq 2\mu(B(\mathcal{S}, \eta_n))$$

for all $n \geq n_1$. Then if $\tilde{f}$ is any continuous extension of $f|M\backslash B(\mathcal{S}, \eta_n)$ to M (which always exists by Tietze Extension Theorem, see e.g. Ref. 42) we get

$$\int |h \circ f - h \circ \tilde{f}| d\mu_n \leq \sup |h| \cdot \mu_n(B(\mathcal{S}, \eta_n)) < \eta \tag{26}$$

for all $n > n_1$. Also note that (26) holds with μ, in the place of μ_n. Since $h \circ \tilde{f}$ is continuous there exists $n_2 > n_1$ such that

$$\left| \int h \circ \tilde{f} d\mu_n - \int h \circ \tilde{f} d\mu \right| < n \quad \text{for every} \quad n > n_2.$$

Hence for $n > n_2$ we get

$$\left| \int h \circ \tilde{f} d\mu_n - \int h \circ \tilde{f} d\mu \right| \leq |\mu(h \circ f) - \mu(h \circ \tilde{f})| + |\mu(h \circ \tilde{f}) - \mu_n(h \circ \tilde{f})|$$
$$+ |\mu_n(h \circ \tilde{f}) - \mu_n(h \circ f)| \leq 3\eta.$$

Since h was an arbitrary continuous function and η was any positive number, we have shown that $f_*\mu_n \to f_*\mu$ in the weak* topology when $n \to \infty$. This is exactly what is needed to show that μ is f-invariant:

$$f_*\mu = \lim_n f_*\mu_n = \lim_n \left(\frac{1}{n} \sum_{j=0}^{n-1} f_*^j \sigma_n + \frac{f_*^n \sigma_n - \sigma_n}{n} \right) = \lim_n \mu_n = \mu,$$

concluding the proof. □

Now we consider $\tilde{J}$ a continuous extension of $J\chi_{M\backslash B(\mathcal{S},P)}$ to M with the same range (this is Tietze's Extension Theorem) for $0 < \rho < \delta$ and write

$$\limsup_{n\to\infty} \mu_n(-J) = \limsup_{n\to\infty} [\mu_n((-J + \tilde{J})\chi_{B(\mathcal{S},\rho)}) + \mu_n(-\tilde{J})]$$

$$\leq 2 \limsup_{n\to\infty} \mu_n(\zeta\,\Delta_\delta) + \mu(-\tilde{J}) \leq 2\zeta\varepsilon - \mu(\tilde{J})$$

since $\tilde{J}$ is continuous and $|-J+\tilde{J}|\chi_{B(\mathcal{S},\rho)} \leq 2|J|\chi_{B(\mathcal{S},\delta)} \leq 2\zeta\,\Delta_\delta$ by (25). Taking $\rho \to 0$ we get $\mu(\tilde{J}) \to \mu(J)$ because $J \in L^1(\mu)$ and together with (21) we arrive at

$$\limsup_{n\to+\infty} \frac{1}{n} \log Z_n \leq h_\mu(f, \mathcal{P}) - \int J d\mu + 2\zeta\varepsilon$$

for some $\mu \in \mathcal{M}_f$ with $\mu(\varphi) \geq c$ and $\Delta_\delta \in L^1(\mu)$, which is enough to prove item (3) of Theorem 4.1 and Theorem A.

5. STRICTLY NEGATIVE UPPER BOUND

Here we prove Theorem B and Theorem D. For a C^1 endomorphism f it is known[50] that the following inequality (also known as *Ruelle's inequality*) holds for every f-invariant probability measure μ

$$h_\mu(f) \leq \int \Sigma^+ d\mu. \tag{27}$$

where Σ^+ denotes the sum of the positive Lyapunov exponents at μ-a.e. point. In Sec. 5.3 we present a proof of this inequality in the setting of maps which are local diffeomorphisms away from a non-flat singular set $\mathcal{S}$ with zero Lebesgue measure, for invariant probability measures μ such that $\log d(x, \mathcal{S})$ is μ-integrable.

We note that in Ref. 33 a similar result was proved under more general geometric assumptions but stricter analytic hypothesis, mostly due to the fact that in Ref. 33 the authors considered M to be a compact metric space admitting a finite dimensional manifold V as an open dense subset and $\mathcal{S} = M\setminus V$, which demands technical conditions on how the Riemannian metric on V and f behave (including the first and second derivatives on local charts) near $\mathcal{S}$ for the proof to work. Our conditions are similar except that we only need the transformation f to be C^1 but assume that $\log d(x, \mathcal{S})$ is integrable, which is natural in our setting.

5.1. The Local Diffeomorphism and Partially Hyperbolic Case

From Ruelle's Inequality (27) and from Sec. 3.3 it follows that we get a non-positive upper bound in item (2) of Theorem A since $\int J d\mu$ equals the sum of the Lyapunov exponents of μ.[44] Moreover let $\mu \in \mathbb{E}$ be given. Then we have

$$\int J d\mu = h_\mu(f) \leq \int \Sigma^+ d\mu \leq \int J d\mu.$$

Hence if $\mu \in \mathcal{M}_f$ is not in $\mathbb{E}$ then the inequality (27) is strict.

To prove Theorem B we fix a continuous $\varphi : M \to \mathbb{R}$ and replace B_n in Sec. 4.1 with

$$B_n = \left\{ x \in M : \inf \left\{ \left| \frac{1}{n} S_n \varphi(x) - \eta(\varphi) \right| : \eta \in \mathbb{E} \right\} > \omega \right\} \qquad (28)$$

for some $\omega > 0$. Then B_n is an open subset of M and we can assume without loss that Leb $(A_n \cap B_n) > 0$ in what follows, for otherwise the limit superior in (8) is smaller than any given real number and there is nothing to prove. Hence arguing as in Sec. 4.1 we obtain a measure $\nu \in \mathcal{M}_f$ satisfying $\inf\{|\nu(\varphi) - \eta(\varphi)| : \eta \in \mathbb{E}\} > \omega$, the bound of item (3) of Theorem A and $\Delta_\delta \in L^1(\nu)$ with $\nu(\Delta_\delta) \leq \varepsilon$.

If f is a local diffeomorphism, i.e. $\mathcal{S} = \emptyset$, then we can use the bound given by item (2) of Theorem A and it is enough to show that $h_\nu(f) - \nu(J)$ is strictly negative. But we cannot have $h_\nu(f) - \nu(J) = 0$ since by construction ν is not in $\mathbb{E}$, thus $h_\nu(f) - \nu(J) < 0$, completing the proof of Theorem B in the case of a local diffeomorphism.

For a partially hyperbolic non-uniformly expanding attracting set we obtain a negative upper bound following the same reasoning as above since we can use the same bound from item (2) of Theorem A, as shown in Sec. 4.2, and we can also apply Ruelle's Inequality. This completes the proof of Theorem D.

5.2. The Case with Singular/Critical Set

In the case $\mathcal{S} \neq \emptyset$ we now show that the upper bound in item (3) of Theorem A must be strictly negative for some values of $\eta, \varepsilon, \delta > 0$ and for some $\nu \in \mathcal{M}_f$. For that we argue by contradiction and take decreasing sequences $\varepsilon_n, \delta_n \to 0$ such that the corresponding measures ν_k obtained according to the proof of Theorem A with B_n as in (28) and

$$A_n^k = \left\{ x \in M : \frac{1}{n} S_n \Delta_{\delta_i} \leq \varepsilon_i, i = 1, \ldots, k \right\}$$

in the place of A_n, for each $k \geq 1$, satisfy

- $\nu_k \in \mathcal{M}_f$, $\Delta_{\delta_i} \in L^1(\nu_k)$ and $\nu_k(\Delta_{\delta_i}) \leq \varepsilon_i$ for $i = 1, \ldots, k$;
- $\limsup_{n\to\infty} \frac{1}{n} \log \text{Leb}\,(A_n^k \cap B_n) \leq h_{\nu_k}(f, \mathcal{P}) - \int J d_{\nu_k} + 2\zeta\varepsilon_k$;
- $h_{\nu_k}(\int, \mathcal{P}) - f J d_{\nu_k} + 2\zeta\varepsilon_k 0$; and
- $\inf\{|\nu_k(\varphi) - \eta(\varphi)| : \eta \in \mathbb{E}\} > \omega$;

where $\mathcal{P}$ is a partition obtained using Lemma 3.4 with the sequence $\mu_k = \nu_k$ and μ some weak* accumulation point of the v_k.

Thus on the one hand we have for any fixed $N \geq 1$

$$h_{\nu_k}(f, \mathcal{P}) = \inf_{j \geq 1} \frac{1}{j} H_{\nu_k} \left(\bigvee_{i=0}^{J-1} f^{-i} \mathcal{P} \right) \leq \frac{1}{N} H_{\nu_k} \left(\bigvee_{i=0}^{N-1} f^{-i} \mathcal{P} \right)$$

and since $\mu(\partial\mathcal{P}) = 0$ we get

$$\limsup_{k\to\infty} h_{\nu_k}(f, \mathcal{P}) \le \frac{1}{N} H_\mu \left(\bigvee_{i=0}^{N-1} f^{-i}\mathcal{P} \right).$$

But $N \ge 1$ was arbitrarily fixed, so

$$\limsup_{k\to\infty} h_{\nu_k}(f, \mathcal{P}) \le \inf_{N\ge 1} \frac{1}{N} H_\mu \left(\bigvee_{i=0}^{N-1} f^{-i}\mathcal{P} \right) = h_\mu(f, \mathcal{P}).$$

On the other hand, choosing J_i to be a continuous extension of $J\chi_{M\setminus B(\mathcal{S},\delta_i)}$ to M with the same range, $i \ge 1$, we have

$$\begin{aligned}\limsup_{k\to\infty} \nu_k(-J) &= \limsup_{k\to\infty} [\nu_k((-J + J_i)\chi_{B(\mathcal{S},\delta_i)}) + \nu_k(-J_i)] \\ &\le 2 \limsup_{k\to\infty} \nu_k(\zeta\,\Delta_{\delta_i}) + \mu(-J_i) \le 2\zeta\,\varepsilon_i - \mu(J_i)\end{aligned}$$

since J_i is continuous and $|-J + J_i|\chi_{B(\mathcal{S},\delta_i)} \le 2|J|\chi_{B(\mathcal{S},\delta_i)} \le 2\zeta\,\Delta_{\delta_i}$ by definition of Δ_{δ_i} and by (25). Similar arguments to the ones proving Lemmas 4.2, 4.3 and 4.4 show that J, ψ, Δ_δ are μ-integrable and that μ is f-invariant. Because $i \ge 1$ can be arbitrarily chosen above and both $\varepsilon_i \to 0$ and $\mu(J_i) \to \mu(J)$, we conclude that $\limsup_{k\to\infty} \nu_k(-J) \le -\mu(J)$. Hence we deduce

$$0 \le \limsup_{k\to\infty} (h_{\nu_k}(f, \mathcal{P}) + \nu_k(-J) + 2\zeta\,\varepsilon_k) \le h_\mu(f, \mathcal{P}) - \mu(J) \le h_\mu(f) - \mu(J)$$

and also that $\inf\{|\mu(\varphi) - \eta(\varphi)| : \eta \in \mathbb{E}\} \ge \omega > 0$ by construction. By Ruelle's Inequality we also get $h_\mu(f) - \mu(J) \le 0$, which yields a contradiction since this means $\mu \in \mathbb{E}$. This contradiction shows that for some $k \ge 1$

$$h_{\nu_k}(f, \mathcal{P}) - \int J\,d\nu_k + 2\zeta\,\varepsilon_k < 0$$

which proves Theorem B, except for the Ruelle Inequality for maps with non-flat singularities, which is the content of the next subsection.

5.3. Ruelle's Inequality for Maps with Non-Flat Singularities

Theorem 5.1. *Let $f : M\setminus\mathcal{S} \to M$ be a C^1 local diffeomorphism away from a non-flat singular set $\mathcal{S}$ and μ a f-invariant probability measure such that $|\log d(x, \mathcal{S})|$ is μ-integrable. Then*

$$h_\mu(f) \le \int \Sigma^+ d\mu,$$

where Σ^+ denotes the sum of the positive Lyapunov exponents at a regular point, counting multiplicities.

Observe that the μ-integrability of $|\log d(x, \mathcal{S})|$ implies the μ-integrability of $\log^+ \|Df\|$, where $\log^+ x = \max\{0, \log x\}$, and thus the Lyapunov exponents of f are well defined μ-almost everywhere by Oseledets Theorem.(44) The proof we present here follows Mañé (Ref. 38, Chap. IV) closely.

We start by taking the M as a compact submanifold of $\mathbb{R}^N$ with the usual Euclidean norm and induced Riemannian structure, and considering W_0 an open *normal tubular neighborhood* of M in $\mathbb{R}^N$, that is, there exists $\Phi : W_0 \to W, (x, u) \mapsto x + u$ a (C^∞) diffeomorphism from a neighborhood W_0 of the zero section of the normal bundle $TM^\perp$ of M to W. Let also $\pi : W \to M$ be the associated projection: $\pi(w)$ is the closest point to w in M for $w \in W$, so that the line through the pair of points $w, \pi(w)$ is normal to M at $\pi(w)$, see e.g. Ref. 31 or Ref. 30. Now we define for $\rho \in (0, 1)$

$$F_0 : W_0 \backslash (T_{\mathcal{S}} M) \to W_0, \quad (x, u) \mapsto (f(x), \rho \cdot u)$$

and also

$$F : W \backslash \Phi(T_{\mathcal{S}} M) \to W, \quad w \mapsto (\Phi \circ F_0 \circ \Phi^{-1})(w).$$

Then clearly F is a local diffeomorphism outside $\Phi(T_{\mathcal{S}} M)$, $\overline{F(W)} \subset W$ and $M = \cap_{n\geq 0} F^n(W)$.

For each $n \geq 1$ consider the partition of $\mathbb{R}^N$ into dyadic cubes

$$\mathcal{P}_n = \left\{ \prod_{i=1}^{N} \left[\frac{a_i}{2^n}, \frac{a_i + 1}{2^n} \right) : a_i \in \mathbb{Z}, i = 1, \ldots, N \right\}.$$

Up to a slight translation of the partitions $\mathcal{P}_n$ we can assume that the probability measure μ on M satisfies $\mu(M \cap \partial\mathcal{P}) = 0$, where $\partial\mathcal{P} = \cup_{n\geq 1} \partial\mathcal{P}_n \cup \mathcal{S}$. For $x \in M \backslash \partial\mathcal{P}$ we define

$$v_n(x) = v_n^F(x) = \#\{P \in \mathcal{P}_n : F(\mathcal{P}n(x)) \cap P \neq \emptyset\}$$

and

$$v(x) = v^F(x) = \limsup_{n\to\infty} v_n(x)$$

where $\mathcal{P}_n(x)$ denotes the atom of the partition $\mathcal{P}_n$ containing x.

Lemma 5.2. *Let $Q = [-1, 1]^N$ and $x \in M \backslash \partial\mathcal{P}$. Then*

$$v(x) \leq \sup_{z\in\mathbb{R}^n} \#\{P \in \mathcal{P}_1 : (z + Dg(x)Q) \cap \neq \emptyset\}$$

Proof: For $x \in M \backslash \partial \mathcal{P}$ and $n \geq 1$ define $\varphi_n(y) = x + y/n$ on $\mathbb{R}^N$ and $W_n = \varphi_n^{-1}(W)$. Let $F_n : W_n \to F_n(W_n) \subset W_n$ be such that

$$\begin{array}{ccc} W_n & \stackrel{F_n}{\to} & W_n \\ \varphi_n \downarrow & & \downarrow \varphi_n \\ W & \stackrel{F}{\to} & W \end{array}$$

commutes. We have $F(w) = F(x) + DF(x)(w - x) + p_x(w)$ where $p_x : W \backslash \Phi(T_S M) \to \mathbb{R}^N$ is C^1 and $\lim_{w \to x} \|p_x(w)\|/\|w - x\| = 0$, where $\| \cdot \|$ is the Euclidean norm on $\mathbb{R}^N$. Then we write $F_n(y) = DF(x)(y) + q_n^x(y) + \alpha_n(x)$ where

$$\alpha_n(x) = n \cdot F(x) - x \quad \text{and} \quad q_n^x(y) = n \cdot p_x(y/n + x). \tag{29}$$

Note that for $x \in M \backslash \partial \mathcal{P}$ we have $q_n^x \to 0$ uniformly on compacta. Indeed if $\|y\| < r$ for some $r > 0$ there is, for each given $\delta > 0$, a $n_0 \in \mathbb{N}$ such that $\|y/n\| < \delta$, $\forall n \geq n_0$ and then, by definition of p_x, for all $\varepsilon > 0$ there is $n_1 \in \mathbb{N}$ so that $\forall n \geq n_1$, $\|p_x(y/n + x)\| < \varepsilon \|y/n\|$ which is the same as $\|n \cdot p_x(y/n + x)\| < \varepsilon r$, or $\|q_n^x(y)\| < \varepsilon r$ for all sufficient large n.

Commutativity of the diagram implies

$$F(\mathcal{P}_n(x)) \cap P \neq \emptyset \Leftrightarrow F_n\left(\varphi_n^{-p}(\mathcal{P}_n(x))\right) \cap \varphi_n^{-1}(P) \neq \emptyset.$$

But $\varphi_n^{-1}(P)$ is an element of $\mathcal{P}_1$ translated by some vector $y_0 \in \mathbb{R}^N$. Moreover $\varphi_n^{-1}(\mathcal{P}_n(x)) \subset Q$ and so $v_n(x) \leq \#\{P \in \mathcal{P}_1 : F_n(Q) \cap (P + y_0) \neq \emptyset\}$. Because α_n depends on x only

$$\begin{aligned} v_n(x) &\leq \#\left\{P \in \mathcal{P}_1 : \left(n \cdot DF(x)\left(\frac{1}{n}Q\right) + q_n^x(Q) + \alpha_n(x) - y_0\right) \cap P \neq \emptyset\right\} \\ &\leq \sup_{z \in \mathbb{R}^N} \#\{P \in \mathcal{P}_1 : (DF(x)Q + q_n^x(Q) + z) \cap P \neq \emptyset\} \end{aligned} \tag{30}$$

Since $q_n^x \to 0$ on compact subsets we get

$$\limsup_{n \to \infty} v_n(x) \leq \sup_{z \in \mathbb{R}^N} \#\{P \in \mathcal{P}_1 : (DF(x)Q + z) \cap P \neq \emptyset\}$$

concluding the proof of the lemma. □

For the arguments which use the convergence properties of the sequence $\log v_n$ we need the following result.

Lemma 5.3. *There exists a μ-integrable function g such that* $0 \leq \log v_n \leq g$ *for μ-almost every point in M and for all $n \geq 1$.*

Proof: Fix $n \geq 1$ and consider $x \in M \backslash \partial \mathcal{P}$. On the one hand since $\mathcal{P}_n$ is a partition we must have $v_n(x) \geq 1$. On the other hand, by the bound (30) since the size of the edge of the cubes of $\mathcal{P}_1$ is 1/2 in $\mathbb{R}^N$ we get

$$v_n(x) \le \big(2(\text{diam } DF(x)(Q) + \text{ diam } q_n^x(Q))\big)^N \tag{31}$$

$$\begin{aligned}\text{diam } DF(x)(Q) &\le 2\sqrt{N} \cdot \|DF(x)\| \\ &\le 2\sqrt{N} \max\{\|Df(x)\|, \|DF|(T_x M)^{\perp}\|\}.\end{aligned} \tag{32}$$

Note that for x far away from $\mathcal{S}$ we always get bounded expressions above since F is a local diffeomorphism outside of $\Phi(T_{\mathcal{S}} M)$. To bound diam $q_n^x(Q)$ we use (29) and consider two cases.

First assume that $d(x, \mathcal{S}) \ge 2/n$ and take $y \in Q$. Then for some $\theta \in [0, 1]$

$$\begin{aligned} q_n^x(y) &= n \cdot p_x(y/n + x) = n \cdot (F(x + y/n) - F(x) - DF(x)(y/n)) \\ &= DF(x + \theta \cdot y/n)(y) - DF(x)(y) \end{aligned}$$

so we get by condition (S1) on $\mathcal{S}$

$$\begin{aligned} \|q_n^x(y) &\le \sqrt{N} \cdot (\|DF(x)\| + \|DF(x + \theta \cdot y/n)\|) \\ &\le B\sqrt{N}(d(x, \mathcal{S})^{-\beta} + (d(x, \mathcal{S}) - 1/n)^{-\beta}) \\ &\le B\sqrt{N} \cdot d(x, \mathcal{S})^{-\beta} \cdot (1 + 2^{\beta}) \end{aligned} \tag{33}$$

since $1 - 1/(nd(x, \mathcal{S})) \ge 1/2$ and $\|DF|(T_x M)^{\perp}\| \le \rho < 1 \ll d(x, \mathcal{S})^{-\beta}$ for x close to $\mathcal{S}$, because $\beta > 0$.

Now assume that $d(x, S) < 2/n$. Then we bound as follows

$$\begin{aligned} \|q_n^x(y)\| &\le n \cdot \|F(x + y/n) - F(x)\| + \|DF(x)\| \cdot \|y\| \\ &\le n \cdot \text{diam } W + B\sqrt{N} \cdot d(x, \mathcal{S})^{-\beta} \end{aligned} \tag{34}$$

Hence putting (31), (32), (33) and (34) together we see that there exists a constant $\tilde{C} > 0$ such that

$$\log v_n(x) \le \left\{ \begin{array}{ll} N \log(\tilde{C} d(x, \mathcal{S})^{-\beta}) & \text{if } d(x, \mathcal{S}) \ge 2/n, \\ N \log(\tilde{C} d(x, \mathcal{S})^{-\beta} + 2n \cdot \text{diam } W) & \text{if } d(x, \mathcal{S}) < 2/n. \end{array} \right\}$$

But $d(x, S)^{-\beta} > 0$ and we may assume without loss that $2n \cdot \text{diam } W \ge 2$, so

$$\log(\tilde{C}(x, \mathcal{S})^{-\beta} + 2n \cdot \text{diam } W) \le \log(\tilde{C} d(x, \mathcal{S})^{-\beta}) + \log(2n \cdot \text{diam } W)$$

and if $d(x, \mathcal{S}) < 2/n$ we also get

$$\begin{aligned} \log d(x, \mathcal{S})^{-\beta} &= -\beta \log d(x, \mathcal{S}) \ge -\beta \log(2/n) = \beta \log(n/2) \\ &= \beta \log(2n \cdot \text{diam } W) - \beta \log(4 \text{ diam } W) \quad \text{or} \end{aligned}$$

$$\log(2n \cdot \text{diam } W) \le \log(4 \text{ diam } W) - \log d(x, \mathcal{S})$$

Hence in all cases we arrive at

$$\log v_n(x) \le N \log \left(C d(x, \mathcal{S})^{-\beta} + D \right)$$

for some positive constants C and D. This concludes the proof. □

Lemma 5.4. *The following bound on the entropy holds*

$$h_\mu(f, \mathcal{P}_n \cap M) = h_\mu(F | \mathcal{P}_n \cap M) \le \int_M \log v_n^F \, d\mu.$$

Proof: This is (Ref. 38, Lemma 12.2) without change. □

Corollary 5.5. $h_\mu(f) = h_\mu(F|M) \le f_M \log v^F d_\mu$.

Proof: Since $\bigvee_{n\ge 1}(\mathcal{P}_n \cap M)$ is the Borel σ-algebra μ mod 0 we get

$$h_\mu(F|M) = \lim_{n\to\infty} h_\mu(F|M, \mathcal{P}_n \cap M) \le \limsup_{n\to\infty} \int_M \log v_n^F \, d\mu.$$

By Lemma 5.3 we can use the Dominated Convergence Theorem to obtain

$$\limsup_{n\to\infty} \int_M \log v_n^F \, d\mu \le \int_M \limsup_{n\to\infty} \log v_n^F \, d\mu = \int_M \log v^F \, d\mu$$

since log is monotonous increasing. This concludes the proof. □

In what follows write $v^n(x) = v^{F^n}(x)$ for the analogous to $v^F(x)$ with F^n in the place of F.

Lemma 5.6. *We have*

$$h_\mu(f) = h_\mu(F|M) \le \int \limsup_{n\to\infty} \frac{1}{n} \log v^n(x) d\mu(x).$$

Proof: Using (Ref. 59, Thm. 4.13) and Corollary 5.5 we get, for all $n \ge 1$

$$h_\mu(F|M) = \frac{1}{n} h_\mu(F^n|M) \le \int \frac{1}{n} \log v^n(x) d\mu(x). \tag{35}$$

Consider the sequence $g_n(x) = n^{-1} \log v^n(x)$ and observe that by Lemma 5.2 and by (32)

$$\begin{aligned} g_n(x) &\le \frac{1}{n} \log(2 \operatorname{diam}(DF^n(x)Q))^N \\ &\le \frac{N}{n} \log(2\sqrt{N}) + \frac{N}{n} \log \|DF^n(x)\| = G_n(x). \end{aligned} \tag{36}$$

Again by (32) and by definition of F since $x \in M$ we get $\log \|DF(x)\| \le \log^+ \|Df(x)\|$. Hence by the f-invariance of μ and the Sub-additive Ergodic Theorem (Ref. 59, Thm. 10.1), the sequence $G_n(x)$ tends to a finite limit $G(x)$ for μ-a.e. x when $n \to \infty$.

Now by (36) and by Fatou's Lemma (Ref. 59, Thm. 0.9)

$$\int \liminf_{n\to\infty}(G_n - g_n)d\mu \le \liminf_{n\to\infty} \int (G_n - g_n)d\mu. \tag{37}$$

On the one hand since $\lim_{n\to\infty} G_n(x)$ exists μ-a.e.

$$\int \liminf_{n\to\infty}(G_n - g_n)d\mu = \int (G - \limsup_{n\to\infty} g_n)d\mu \tag{38}$$

and, on the other hand, since $\lim_{n\to\infty} \int G_n(x)d\mu$ exists μ-a.e. we also get

$$\liminf_{n\to\infty} \int (G_n - g_n)d\mu = \int G d\mu - \limsup_{n\to\infty} \int g_n d\mu. \tag{39}$$

Altogether (37), (38) and (39) imply

$$\limsup_{n\to\infty} \int \frac{1}{n} \log v^n(x) d\mu(x) \le \int \limsup_{n\to\infty} \frac{1}{n} \log v^n(x) d\mu(x)$$

which together with (35) conclude the proof of the Lemma. □

To finish we need to relate $\limsup_{n\to\infty} \frac{1}{n} \log v^n(x)$ with the sum of the positive Lyapunov exponents at x. This is done just as in (Ref. 38, Chap. IV, Sec. 12) where it is proved that

$$\limsup_{n\to\infty} \frac{1}{n} \log v^n(x) \le \Sigma^+(x)$$

for μ-almost all $x \in M$. This together with Lemma 5.6 implies Ruelle's Inequality. The proof of Theorem 5.1 is complete.

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